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Abstract

This project, termed *Cell Sets and Decomposition Spaces*, is an introduction to the connections between homotopy and algebraic combinatorics. The main goal is to understand the category **Cell** of cell-sets and cell-maps, introduced by N. Ray and W. Schmitt, in the more modern language of 2-Segal spaces, also known as decomposition spaces. Recent work on decomposition spaces has shown them to provide a powerful language for combinatorial structures and their symmetries, and it is illuminating to see the examples and constructions of Ray–Schmitt from this perspective.

In order to do so we construct a functor between the category of (integer graded, discrete) decomposition spaces and **Cell**, which factors through the bicategory of spans of (integer graded, discrete) groupoids by identifying the cell-sets and cell-maps of Ray–Schmitt as discrete groupoids over $\mathbb{Z}$ and spans or correspondences between them, respectively. We use it, together with the free abelian group functor defined by Ray–Schmitt in their work, to connect decomposition spaces with graded coalgebras and, more generally, graded Hopf algebras.

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Chapter 0

Introduction

The theory of decomposition spaces was first introduced by Dyckerhoff and Kapranov in [DK19] under the name of unital 2-Segal space. Independently, and starting from a very different motivation, Gálvez-Carrillo, Kock and Tonks arrived at an equivalent notion (see [GCKT18a], [GCKT18b] and [GCKT18c]), which is the one termed decomposition space.

Decomposition spaces are simplicial ∞-groupoids verifying the following exactness condition: active-inert (or generic-free) pushout squares in the simplex category Δ are sent to homotopy pullback squares of ∞-groupoids. This technical condition is demanded in order to induce a (strong) comonoid object in the symmetric monoidal ∞-category **LIN**, consisting of homotopy slices of ∞-groupoids and linear functors between them (see [GCKT18d] for a deeper understanding of **LIN**), and in fact such a condition is essentially an equivalence. Therefore, decomposition spaces encode, in some sense, the minimal required information such that the coassociative coalgebra structure is achieved. At first glance, this definition looks far different from the concept of 2-unital Segal space: the definition of decomposition space is based on preservation of certain pullbacks, whereas the definition of unital 2-Segal space refers to triangulations of convex polygons. But it turns out that, as a consequence of the fact that some basic results concerning these notions coincide, it was noticed that indeed both concepts are the same. Concretely, the results are the characterization of decomposition spaces in terms of decalage and Segal spaces (see [GCKT18a], theorem 4.10) and the result that asserts that the Waldhausen $S_\bullet$-construction of a stable ∞-category is a decomposition space (see [GCKT18a], theorem 10.15), which were obtained first and independently in [DK19].

On the other hand, inspired by the notion of CW-complex and mainly motivated by applications in algebraic topology, Ray and Schmitt introduce in [RS98] a category **Cell** of combinatorial objects, known as cell-sets, whose morphisms are known as cell-maps. They equip **Cell** with a suitable symmetric monoidal structure, and define a (strong) monoidal functor $Z_*\colon \mathbf{Cell} \longrightarrow \mathbf{GAb}$ named the free abelian group functor. Hence, such combinatorial objects are used to produce, among other algebraic objects, classical algebras, coalgebras, bialgebras and Hopf algebras in the category of graded abelian groups **GAb** by previously defining such structures at the level of cell-sets.

That said, the main goal of this thesis is to establish a functorial connection between decomposition spaces and the category **Cell** of cell-sets and cell-maps, in such a way that the algebraic constructions developed in [RS98] are reinterpreted in terms of decomposition spaces. Cell-sets are sets with some specified equivalence relation and a compatible dimension function. In the same way that preorders are easily identified with discrete categories, equivalence relations are just discrete groupoids, since the symmetry condition allows us to invert the arrows. Therefore, a cell-set is just a discrete groupoid G together with a dimension functor $D\colon G \longrightarrow \mathbb{Z}$. The reinterpretation of morphisms is a bit more subtle: cell-maps are assignments $f\colon \mathscr{C} \longrightarrow \mathscr{D}$ that associate, to every $x \in \mathscr{C}$, a finite submultiset of the cell-set $\mathscr{D}$ in such a way that some preserving conditions are satisfied. The problem is that the authors of [RS98] do not specify in detail how these conditions work, nor they give an explicit definition of their notion of submultiset. In order to fix this issue, we provide a notion of submultiset that we think that fits better in their framework. With such a notion, cell-maps are seen as maps taking values at slices of sets. This key fact suggests us the possibility of defining cell-maps starting from spans of discrete groupoids over the integers, essentially in the same way as spans of ∞-groupoids define linear functors between the homotopy slices. With this philosophy in mind, we introduce a certain category of spans of discrete groupoids over $\mathbb{Z}$

$$\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$$

with some specified conditions. We equip this category with an appropriate symmetric monoidal structure and we construct a (strict) monoidal functor

$$\Psi\colon \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \longrightarrow \mathbf{Cell},$$

termed cellularization functor, such that the triangle

$$\begin{tikzcd}
\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \arrow[r, "\Psi"] \arrow[dr, "\|\cdot\|"'] & \mathbf{Cell} \arrow[d, "Z_*"] \\
& \mathbf{GAb}
\end{tikzcd}$$

commutes, where

$$\|.\| : \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \longrightarrow \mathbf{Cell}$$

is the cardinality functor defined in [GCKT18d] but adapted to our particular case of discrete 1-groupoids. Once this construction is done, we can introduce the role of decomposition spaces: we work with a modified category of decomposition spaces, named graded discrete decomposition spaces and denoted by **GrDecomp**, which consists essentially of simplicial discrete groupoids over $\mathbb{Z}$, verifying the decomposition-space axiom, and CULF simplicial maps between them. We equip the slice

$$\mathbf{DiscGrpd}/\mathbb{Z}$$

with a symmetric monoidal structure very similar to the one defined in the category of spans, and then we mimic the construction of the incidence coalgebra of a decomposition space (done, for instance, in [GCKT18a], section 5), using the simplicial axioms to construct a comonoid object in $\overline{\textbf{PSpan}}(\textbf{DiscGrpd}/\mathbb{Z})$. Furthermore, the monoidal structure of $\textbf{DiscGrpd}/\mathbb{Z}$ is seen to provide a monoidal structure in **GrDecomp** (defined pointwise), allowing us to talk about monoidal decomposition spaces and CULF monoidal functors. With this extra structure, which does not come from the internal simplicial structure of decomposition spaces, we naturally induce bimonoid objects within our category of spans, as it is analogously done in [GCKT18a], section 9. Thus, we have a chain of functors

$$\textbf{GrDecomp} \xrightarrow{\Phi} \overline{\textbf{PSpan}}(\textbf{DiscGrpd}/\mathbb{Z}) \xrightarrow{\Psi} \textbf{Cell} \xrightarrow{Z_*} \textbf{GAb}$$

that sends graded discrete decomposition spaces and monoidal decomposition spaces to coalgebras and bialgebras in **GAb**, respectively, and similarly with CULF and monoidal CULF functors.

Finally, let us conclude the introduction by commenting how this thesis is structured, giving a brief but detailed explanation of the things we have done in each chapter.

Chapter 1 In the first chapter, we first give the most fundamental definitions concerning the classical theory of algebras, coalgebras, bialgebras and Hopf algebras in the monoidal category of vector spaces. After this first contact with the classical theory, we jump into the more general notion of monoidal and symmetric monoidal categories, together with their corresponding monoid, comonoid, bimonoid and Hopf monoid objects. We also describe the appropriate notions of morphism of monoidal categories, namely lax, colax and bilax monoidal functors, which are in some sense the most affordable morphisms that one can have in order to preserve monoid, comonoid and bimonoid objects, respectively. Finally, we briefly talk about bicategories since we need this notion to construct the category of spans $\overline{\textbf{PSpan}}(\textbf{DiscGrpd}/\mathbb{Z})$.

Chapter 2 Once we have established what we consider preliminary background, in the second chapter we study in detail the category **Cell** of cell-sets and cell-maps. As we said, the authors of [RS98] do not provide any definition of submultiset, a fact that has made it difficult for us to make progress in our work. Of course, everyone understands the naive definition of submultiset as a set where we are allowed to repeat elements, but the way we encode these multiplicities is very important in the context of cell-sets. We first based our definition in the work of Monro done in [Mon87], but their definitions turned out to be useless for the context of cell-sets. Thus, we finally decided to define, given a set A, its set of submultisets as an isomorphism class in the slice category $\textbf{Set}/A$. As we said above, this definition has played an important role in the definition of the cellularization functor.

With our fitting notion of submultiset, we find mathematical inconsistencies in the work done in [RS98], which we remark during the chapter. Examples of such inconsistencies are, for instance, objects that are claimed to be terminal under wrong conditions, the inconsistent definition of pointed cell-map and, as a consequence of these errors, the wrong sufficient conditions to obtain algebraic structures in **Cell**.

In the last two sections, we introduce an important class of cell-sets coming from interval categories, which are essentially categories where the objects are intervals from a given poset, equipped with a generalization of the notion of rank function.

Chapter 3 In the third chapter we develop the theory which constitutes the main part of the original work that has been done in this thesis. Here we construct the cellularization functor and, furthermore, we prove that it is full and bijective on objects, but regrettably it fails to be faithful, as the constructive proof of the fullness shows. We also provide the category of spans $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$ with a symmetric monoidal structure in such a way that the cellularization functor becomes a strict monoidal functor. Together with the free abelian group functor, this means that we have a way to build algebraic structures at the level of spans, instead of at the level of cell-sets, that can be transported to **GAb**.

Chapter 4 Now that we have developed the functorial relationship between

$$\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \text{ and } \mathbf{Cell},$$

we just need to proceed as in the work done with decomposition spaces and the ∞-category **LIN**. The only difference is that now we need to work in a different category of decomposition spaces. As we said above, we are forced to deal with simplicial discrete groupoids over $\mathbb{Z}$ with the decomposition-space condition and CULF simplicial maps. As we also explained earlier, we show how these decomposition spaces induce comonoid objects in the category of spans and, considering the monoidal structure induced in **GrDecomp**, we show how we immediately have bimonoid objects in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$.

For lack of time, we have not been able to rewrite a large class of examples of [RS98] into the language of decomposition spaces. However, we close the last chapter by giving an example of an interval cell-set with its associated decomposition space. We hope that, in a future work, such purpose will be achieved.

Chapter 1

Monoidal Structures

Roughly speaking, a bialgebra is an algebra in the classical sense together with a dual structure, called coalgebra, such that both structures verify a compatibility condition. A Hopf algebra is a bialgebra which, in addition, comes equipped with a linear endomorphism verifying another compatibility condition expressible in terms of the bialgebra structure.

Hopf algebras were first observed by Heinz Hopf in 1941 while studying algebraic topology, concretely the homology of a connected Lie group. Later on, Hopf algebras were studied for its own sake within an algebraic framework, and around the end of the '80s the subject started to have connections with a wide variety of areas, some seemingly distant such as quantum mechanics.

In this chapter we first review the most basic definitions related to the theory of Hopf algebras. Then we go into the more general concept of monoidal category, which is a generalization of the tensor product in more general categories that provides a framework where, among other things, we can generalize the classical structures of algebra, coalgebra, bialgebra and Hopf algebra. Finally, we conclude by briefly analyzing the notion of bicategory, or weak 2-category, which will be needed essentially to talk about the bicategory of spans in the subsequent chapters.

1.1 Classical Hopf Algebras

Throughout this section, $\mathbb{K}$ will be a field, and all the non-labeled tensor products will be over $\mathbb{K}$. The same theory can be developed in an analogous way replacing vector spaces by modules over a commutative ring R. The main reference is [DNR01], which is a suitable reference for those who want to go deeper into the subject for the first time. We omit lots of interesting parts such as the linear dual of a (co)algebra, due to the fact that we did not have time to apply these notions in our work.

Definition 1.1.1. A $\mathbb{K}$-algebra is a triple (A, μ, η), where A is a $\mathbb{K}$-vector space, $\mu\colon A \otimes A \longrightarrow A$ and $\eta\colon \mathbb{K} \longrightarrow A$ are $\mathbb{K}$-linear maps called *multiplication* and *unit*, respectively, such that the following diagrams commute

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}_A \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}_A"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dd, "\mu"] & \\
\mathbb{K} \otimes A \arrow[ur, "\eta \otimes \mathrm{id}_A"] \arrow[dr, "\lambda_A"'] & & A \otimes \mathbb{K}, \arrow[ul, "\mathrm{id}_A \otimes \eta"'] \arrow[dl, "\rho_A"] \\
 & A &
\end{tikzcd}
$$

where

$$
\begin{array}{rcl}
\lambda_A \colon \mathbb{K} \otimes A & \longrightarrow & A \\
1 \otimes x & \longmapsto & x
\end{array}
\qquad
\begin{array}{rcl}
\rho_A \colon A \otimes \mathbb{K} & \longrightarrow & A \\
x \otimes 1 & \longmapsto & x
\end{array}
$$

are the canonical isomorphisms. The commutativity of the first diagram is called *associativity* and the commutativity of the second is called *the unit property.*

It is straightforward to show that this arrows-only definition is the usual definition of a $\mathbb{K}$-algebra of vector spaces. One of the advantages of defining it this way is that it can be dualized in the sense of the opposite category, that is, reversing the arrows.

Definition 1.1.2. A $\mathbb{K}$-coalgebra is a triple (C, δ, ε), where C is a $\mathbb{K}$-vector space, $\delta \colon C \longrightarrow C \otimes C$ and $\varepsilon \colon C \longrightarrow \mathbb{K}$ are $\mathbb{K}$-linear maps called *comultiplication* and *counit*, respectively, such that the following diagrams are commutative

$$
\begin{tikzcd}
C \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes C \arrow[d, "\mathrm{id}_C \otimes \delta"] \\
C \otimes C \arrow[r, "\delta \otimes \mathrm{id}_C"'] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \arrow[dl, "\lambda_C^{-1}"'] \arrow[dr, "\rho_C^{-1}"] \arrow[dd, "\delta"] & \\
\mathbb{K} \otimes C & & C \otimes \mathbb{K}. \\
 & C \otimes C \arrow[ul, "\varepsilon \otimes \mathrm{id}_C"] \arrow[ur, "\mathrm{id}_C \otimes \varepsilon"'] &
\end{tikzcd}
$$

The commutativity of the first diagram is called *coassociativity* and the commutativity of the second is called *the counit property.*

Since this definition is not that common, we give some basic examples.

Examples 1.1.3. *1. Any vector space can be equipped with a coalgebra structure: indeed, let $F(S)$ be the free $\mathbb{K}$-vector space with basis S. Then, $F(S)$ is a coalgebra with structure maps*

$$
\begin{array}{rcl}
\delta \colon F(S) & \longrightarrow & F(S) \otimes F(S) \\
s & \longmapsto & s \otimes s
\end{array}
\qquad
\begin{array}{rcl}
\varepsilon \colon F(S) & \longrightarrow & \mathbb{K} \\
s & \longmapsto & 1.
\end{array}
$$

Suppose now that $S = \{s_n \colon n \in \mathbb{N}\}$. Then, the maps

$$
\begin{array}{rcl}
\delta \colon F(S) & \longrightarrow & F(S) \otimes F(S) \\
s_n & \longmapsto & \sum_{i=0}^{n} s_i \otimes s_{n-i}
\end{array}
\qquad
\begin{array}{rcl}
\varepsilon \colon F(S) & \longrightarrow & \mathbb{K} \\
s_n & \longmapsto & \delta_{0,n}
\end{array}
$$

define a coalgebra structure on $F(S)$, called the divided power coalgebra, where $\delta_{i,j}$ is the Kronecker symbol.

2. *Let $(P, \preceq)$ be a partially ordered set (poset) which is locally finite, that is, every interval*
$$[x, y] = \{z \in P : x \preceq z \preceq y\}$$
has a finite number of elements. Let
$$T = \{(x, y) \in P \times P \colon x \preceq y\}$$
and let $F(T)$ be the free $\mathbb{K}$-vector space on the set T. Then, $F(T)$ is a coalgebra, called the incidence coalgebra of P, with structure maps given by
$$\begin{array}{rcl} \delta\colon F(T) & \longrightarrow & F(T) \otimes F(T) \\ (x, y) & \longmapsto & \sum_{x \preceq z \preceq y} (x, z) \otimes (z, y) \end{array} \qquad \begin{array}{rcl} \varepsilon\colon F(T) & \longrightarrow & \mathbb{K} \\ (x, y) & \longmapsto & \delta_{x,y}. \end{array}$$

The ordinary commutativity condition can be rewritten in terms of a commutative diagram, and therefore we also have its dual notion, which expresses commutativity in the dual structure.

Definition 1.1.4. An algebra (A, μ, η) is said to be *commutative* if the following diagram commutes
$$\begin{tikzcd} A \otimes A \arrow[rr, "T_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A, & \end{tikzcd}$$
where, given two vector spaces A and B, $T_{A,B} \colon A \otimes B \longrightarrow B \otimes A$ is the *twist map* defined by the linear map
$$\begin{array}{rcl} T = T_{A,B} \colon A \otimes B & \longrightarrow & B \otimes A \\ x \otimes y & \longmapsto & y \otimes x. \end{array}$$
Dually, a coalgebra (C, δ, ε) is called *cocommutative* if the diagram
$$\begin{tikzcd} & C \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ C \otimes C \arrow[rr, "T_{C,C}"'] & & C \otimes C \end{tikzcd}$$
is commutative.

Definition 1.1.5. Let (A, μ_A, η_A) and (B, μ_B, η_B) be two $\mathbb{K}$-algebras and let $f \colon A \longrightarrow B$ be a morphism of $\mathbb{K}$-vector spaces. The linear map f is said to be a morphism of $\mathbb{K}$-algebras if the following diagrams are commutative
$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] & & B \\ & \mathbb{K}. \arrow[ul, "\eta_A"] \arrow[ur, "\eta_B"'] & \end{tikzcd}$$

Dually, let $(C, \delta_C, \varepsilon_C)$ and $(D, \delta_D, \varepsilon_D)$ be two $\mathbb{K}$-coalgebras and let $g\colon C \longrightarrow D$ be a $\mathbb{K}$-linear map. We say that g is a morphism of $\mathbb{K}$-coalgebras if the following diagrams are commutative

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\delta_C"'] & D \arrow[d, "\delta_D"] \\ C \otimes C \arrow[r, "g\otimes g"'] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "g"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & \mathbb{K}. & \end{tikzcd}$$

In this way, we may construct the categories $\mathbf{Mon}(\mathbf{Vect}_{\mathbb{K}})$ and $\mathbf{Comon}(\mathbf{Vect}_{\mathbb{K}})$ consisting of $\mathbb{K}$-algebras and $\mathbb{K}$-coalgebras, together with their corresponding morphisms between them, respectively. The notation used to describe such categories will be justified in the next section.

We now jump into the notion of bialgebra. Consider a $\mathbb{K}$-vector space which has both the structure of an algebra (H, μ, η) and a coalgebra (H, δ, ε). In this case, the product $H \otimes H$ has a natural induced structure of algebra $(H \otimes H, \mu', \eta')$ and coalgebra $(H \otimes H, \delta', \varepsilon')$ given by the maps

$$\mu' = (\mu \otimes \mu) \circ (\mathrm{id}_H \otimes B_{H,H} \otimes \mathrm{id}_H), \;\; \eta' = \mathbb{K} \cong \mathbb{K} \otimes \mathbb{K} \xrightarrow{\eta\otimes\eta} H \otimes H,$$

$$\delta' = (\mathrm{id}_H \otimes B_{H,H} \otimes \mathrm{id}_H) \circ (\delta \otimes \delta) \text{ and } \;\; \varepsilon' = H \otimes H \xrightarrow{\varepsilon\otimes\varepsilon} \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}.$$

Therefore, it makes sense to ask if μ and η are coalgebra morphisms or if δ and ε are algebra morphisms. Playing around with the diagrams, it is not hard to see that in fact both conditions are equivalent, that is,

$$\mu \text{ and } \eta \text{ are morphisms of coalgebras} \iff \delta \text{ and } \varepsilon \text{ are morphisms of algebras.}$$

If one of these equivalent conditions hold, we obtain the notion of bialgebra.

Definition 1.1.6. A *bialgebra* is a $\mathbb{K}$-vector space H endowed with an algebra structure (H, μ, η) and a coalgebra structure (H, δ, ε) such that both structures are compatible, that is, one of the above-mentioned equivalent conditions hold.

Definition 1.1.7. Let H and L be two $\mathbb{K}$-bialgebras. A $\mathbb{K}$-linear map $f\colon H \longrightarrow L$ is called a morphism of bialgebras if it is both a morphism of algebras and coalgebras between the underlying algebras and coalgebras.

Thus, there is a category $\mathbf{Bimon}(\mathbf{Vect}_{\mathbb{K}})$ consisting of bialgebras and bialgebra morphisms.

Finally, we state the last compatibility condition needed in order to construct a Hopf algebra. Let $(H, \mu, \eta, \delta, \varepsilon)$ be a bialgebra. We can endow the vector space $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(H, H)$ of linear endomorphisms with an algebra structure by defining the *convolution product*

$$\begin{array}{rrcl} *\colon & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(H,H) \times \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(H,H) & \longrightarrow & \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(H,H) \\ & (f,g) & \longmapsto & f * g := \mu \circ (f \otimes g) \circ \delta, \end{array}$$

that is, $f * g$ is given by the composition of the maps

$$H \xrightarrow{\delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H.$$

Thanks to the bialgebra structure, it can be seen that the convolution product is associative, with unit

$$\eta \circ \varepsilon \colon H \longrightarrow H \in \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(H, H).$$

In general, $\eta \circ \varepsilon \neq \mathrm{id}_H$. Hence, the existence of the convolution inverse of the identity map $\mathrm{id}_H \in \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(H, H)$ is not guaranteed in a general bialgebra H. Its existence is precisely the compatibility condition that defines a Hopf algebra.

Definition 1.1.8. Let $(H, \mu, \eta, \delta, \varepsilon)$ be a bialgebra. A linear map $S \colon H \longrightarrow H$ is called an *antipode* of H if it is the convolution inverse of the identity map $\mathrm{id}_H \in \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(H, H)$. If H has an antipode, we say that H is a *Hopf algebra*.

Definition 1.1.9. Let H and H' be two Hopf algebras. A map $f \colon H \longrightarrow H'$ is called a morphism of Hopf algebras if it is a morphism of bialgebras.

One could wonder why do not we add the condition of preserving the antipodes, but it turns out that if $f \colon (H, S) \longrightarrow (H', S')$ is a bialgebra morphism, then we have that the diagram

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "S"'] & H' \arrow[d, "S'"] \\ H \arrow[r, "f"'] & H' \end{tikzcd}$$

is commutative. This fact is easily proven by showing that both $S' \circ f$ and $f \circ S$ are convolution-invertible, sharing the same inverse f.

1.2 Monoidal Categories and Bicategories

The purpose of this section is to give a few definitions related to the concepts of monoidal categories and bicategories, since perhaps these topics are not that usual in a basic course in ordinary category theory. Anyway, the concept of monoidal category, (co)monoid object and related topics play a key role throughout the work, as well as the concept of bicategory with the bicategory of spans. Therefore, it is convenient to get familiar with them if one wants to understand the work we have done. Furthermore, even though the reader may be familiar with the concepts, it is also a good idea for the sake of terminology. Our main references here are the following ones: [Mac71], for the basic notions of monoidal categories, [Por15], for the notions of monoid object and related concepts, [AM10] for the background in monoidal functors and [JY20] for the definition of bicategory.

1.2.1 Monoidal Categories

A monoidal category is a category $\mathcal{C}$ endowed with some sort of 'tensor product' $\otimes$, given by a bifunctor

$$\otimes\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}.$$

The canonical example is the category $\mathbf{Vect}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$ together with linear maps. Universality of the tensor product not only allows us to compute tensor products of vector spaces but also of linear maps. Another natural example is the category **Set** together with the cartesian product or disjoint union. In particular, a category can be made into a monoidal category in more than one way. The formal definition is the following.

Definition 1.2.1. A *monoidal category* is a category $\mathcal{C}$ together with the following data:

1. A bifunctor
$$\otimes\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$
called the *tensor product*.

2. An object
$$I \in \mathcal{C}$$
called the *unit object*.

3. A natural isomorphism
$$a\colon ((-) \otimes (-)) \otimes (-) \longrightarrow (-) \otimes ((-) \otimes (-))$$
with components, for every $x, y, z \in \mathcal{C}$, of the form
$$a_{x,y,z}\colon (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$
called the *associator*.

4. A pair of natural isomorphisms
$$\lambda\colon I \otimes (-) \longrightarrow (-), \quad \rho\colon (-) \otimes I \longrightarrow (-)$$
with components, for every $x \in \mathcal{C}$, of the form
$$\lambda_x\colon I \otimes x \longrightarrow x, \quad \rho_x\colon x \otimes I \longrightarrow x.$$
They are called the *left unitor* and the *right unitor*, respectively.

Moreover, for every $x, y, z, t \in \mathcal{C}$, the following diagrams are commutative:

1. *Triangle identity*:

$$\begin{tikzcd}
(x \otimes I) \otimes y \arrow[rr, "a_{x,I,y}"] \arrow[dr, "\rho_x \otimes \mathrm{id}_y"'] & & x \otimes (I \otimes y) \arrow[dl, "\mathrm{id}_x \otimes \lambda_y"] \\
& x \otimes y. &
\end{tikzcd}$$

2. *Pentagon identity*:

$$
\begin{tikzcd}
 & (t \otimes x) \otimes (y \otimes z) \arrow[dr, "a_{t,x,y\otimes z}"] & \\
((t \otimes x) \otimes y) \otimes z \arrow[ur, "a_{t\otimes x,y,z}"] \arrow[d, "a_{t,x,y}\otimes \mathrm{id}_z"'] & & t \otimes (x \otimes (y \otimes z)) \\
(t \otimes (x \otimes y)) \otimes z \arrow[rr, "a_{t,x\otimes y,z}"'] & & t \otimes ((x \otimes y) \otimes z). \arrow[u, "\mathrm{id}_t \otimes a_{x,y,z}"']
\end{tikzcd}
$$

3. *Trivial identity*:

$$\lambda_I = \rho_I \colon I \otimes I \longrightarrow I.$$

We say that $\mathcal{C}$ is a *symmetric monoidal category* if, in addition, there exists a natural isomorphism

$$B \colon (-) \otimes (-) \longrightarrow (-) \otimes (-)$$

with components, for every $x, y \in \mathcal{C}$, of the form

$$B_{x,y} \colon x \otimes y \longrightarrow y \otimes x$$

called the *braiding*. We also demand, for every $x, y, z \in \mathcal{C}$, that the following diagrams are commutative:

1. *Hexagon identity*:

$$
\begin{tikzcd}
 & x \otimes (y \otimes z) \arrow[dr, "B_{x,y\otimes z}"] & \\
(x \otimes y) \otimes z \arrow[ur, "a_{x,y,z}"] \arrow[d, "B_{x,y}\otimes \mathrm{id}_z"'] & & (y \otimes z) \otimes x \arrow[d, "a_{y,z,x}"] \\
(y \otimes x) \otimes z \arrow[dr, "a_{y,x,z}"'] & & y \otimes (z \otimes x) \\
 & y \otimes (x \otimes z). \arrow[ur, "\mathrm{id}_y \otimes B_{x,z}"'] &
\end{tikzcd}
$$

2. *Inverse identity*:

$$
\begin{tikzcd}
x \otimes y \arrow[rr, equal] \arrow[dr, "B_{x,y}"'] & & x \otimes y \\
 & y \otimes x \arrow[ur, "B_{y,x}"'] &
\end{tikzcd}
$$

3. *Unit identity*:

$$
\begin{tikzcd}
x \otimes I \arrow[rr, "B_{x,I}"] \arrow[dr, "\rho_x"'] & & I \otimes x \arrow[dl, "\lambda_x"] \\
 & x. &
\end{tikzcd}
$$

One of the most fundamental examples are categories with finite products. They are naturally symmetric monoidal categories, with binary product as tensor product and terminal object as unit object. We call them *cartesian monoidal categories*. There is a dual notion, called *cocartesian monoidal category*, defined in an analogous way. In particular, the category **Set** with cartesian product as tensor product is the simplest example of cartesian monoidal category.

There is also a notion of *strict monoidal category*, which is the same but demanding that the associator, left unitor and right unitors are all identities. However, in practice this kind of categories do not occur very frequently.

It can be shown that the axioms demanded in a monoidal category are enough to ensure that 'every' diagram involving the associator, the left unitor and the right unitor is commutative. For a more detailed explanation, see [Mac71], **VII**.2. For the case of symmetric monoidal categories there is an extension of such coherence theorem (see [Mac71], **XI**.1).

Recall that a monoid is a set M equipped with a binary operation $\mu\colon M \times M \longrightarrow M$ and a distinguished element $e \in M$, termed the neutral element, verifying the usual associativity and unit laws. Equivalently, monoids are in one-to-one correspondence with categories with one object in a natural way.

The purpose of the following definition is to generalize monoids in **Set** to any monoidal category $(\mathcal{C}, \otimes, I)$ in such a way that, in particular, we can recover the notion of classical monoid when considering the category **Set** with the cartesian monoidal structure. It may also be seen as the generalization of an algebra in $\mathbf{Vect}_{\mathbb{K}}$ (see 1.1).

Definition 1.2.2. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A *monoid object* in $\mathcal{C}$ is an object $M \in \mathcal{C}$ together with a morphism called *multiplication*

$$\mu\colon M \otimes M \longrightarrow M$$

and a morphism called *unit*

$$\eta\colon I \longrightarrow M$$

such that the following axioms are satisfied:

1. *Associative law*:

$$\begin{tikzcd}
 & M & \\
M \otimes M \arrow[ur, "\mu"] & & M \otimes M \arrow[ul, "\mu"'] \\
(M \otimes M) \otimes M \arrow[u, "\mu \otimes \mathrm{id}_M"] \arrow[rr, "a_{M,M,M}"'] & & M \otimes (M \otimes M). \arrow[u, "\mathrm{id}_M \otimes \mu"']
\end{tikzcd}$$

2. *Left and right unit laws*:

$$\begin{tikzcd} I \otimes M \arrow[r, "\eta\otimes \mathrm{id}_M"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "\mu"] & M \otimes I \arrow[l, "\mathrm{id}_M \otimes \mu"'] \arrow[dl, "\rho_M"] \\ & M. & \end{tikzcd}$$

In the definiton a is the associator, while λ and ρ are the left and right unitors, respectively. Moreover, if $\mathcal{C}$ has the structure of symmetric monoidal category $(\mathcal{C}, \otimes, I, B)$, a monoid object (M, μ, η) is termed *commutative monoid in* $(\mathcal{C}, \otimes, I, B)$ if, in addition, the following diagram is commutative

$$\begin{tikzcd} M \otimes M \arrow[rr, "B_{M,M}"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M. & \end{tikzcd}$$

Analogous to the classical concept of morphism of monoids, we have the more general notion of monoid morphism in any monoidal category.

Definition 1.2.3. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and let (M, μ, η) and (M', μ', η') be a pair of monoid objects in $\mathcal{C}$. A *morphism of monoids*

$$f : (M, \mu, \eta) \longrightarrow (M', \mu', \eta')$$

is a morphism $f : M \longrightarrow M'$ in $\mathcal{C}$ such that it is *multiplicative* and *unital*, which means that the following diagrams

$$\begin{tikzcd} M \otimes M \arrow[r, "f\otimes f"] \arrow[d, "\mu"'] & M' \otimes M' \arrow[d, "\mu'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "f"] & & M' \\ & I \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

are commutative, respectively.

As one can easily check, given a monoidal category, monoid objects and monoid morphisms constitute a category.

Definition 1.2.4. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. The *category of monoids in* $\mathcal{C}$, which we denote by $\mathbf{Mon}(\mathcal{C})$, is the category consisting of monoid objects in $\mathcal{C}$ and monoid morphisms between them. If, moreover, $\mathcal{C}$ is a symmetric monoidal category $(\mathcal{C}, \otimes, I, B)$, we denote by $\mathbf{Mon_c}(\mathcal{C})$ the full subcategory of $\mathbf{Mon}(\mathcal{C})$ consisting of commutative monoids.

Dualizing the previous notions related to monoidal categories, we obtain the concepts of comonoid object and comonoid morphism, and therefore the category of comonoid objects. Due to its importance in our work, and for the sake of completeness, we write their definitions in detail.

Definition 1.2.5. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A *comonoid object in* $\mathcal{C}$ is a monoid object C in the opposite category $\mathcal{C}^{op}$. This means that C comes equipped with a morphism called *comultiplication*

$$\delta\colon C \longrightarrow C \otimes C$$

and a morphism called *counit*

$$\varepsilon\colon C \longrightarrow I$$

such that the following axioms hold:

1. *Coassociative law*:

$$\begin{tikzcd} & C \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ C \otimes C \arrow[d, "\delta\otimes\mathrm{id}_C"'] & & C \otimes C \arrow[d, "\mathrm{id}_C\otimes\delta"] \\ (C \otimes C) \otimes C \arrow[rr, "a_{C,C,C}"'] & & C \otimes (C \otimes C). \end{tikzcd}$$

2. *Left and right counit laws*:

$$\begin{tikzcd} I \otimes C \arrow[dr, "\lambda_C"'] & C \otimes C \arrow[l, "\varepsilon\otimes\mathrm{id}_C"'] \arrow[r, "\mathrm{id}_C\otimes\varepsilon"] & C \otimes I \arrow[dl, "\rho_C"] \\ & C. \arrow[u, "\delta"] & \end{tikzcd}$$

As before, if $\mathcal{C}$ has the structure of symmetric monoidal category $(\mathcal{C}, \otimes, I, B)$, a comonoid object (C, δ, ε) is named *cocommutative comonoid in* $(\mathcal{C}, \otimes, I, B)$ if, moreover, the following diagram is commutative

$$\begin{tikzcd} C \otimes C \arrow[rr, "B_{C,C}"] & & C \otimes C \\ & C. \arrow[ul, "\delta"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

Definition 1.2.6. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and let (C, δ, ε) and $(C', \delta', \varepsilon')$ be a pair of comonoid objects in $\mathcal{C}$. A *morphism of comonoids*

$$f\colon (C, \delta, \varepsilon) \longrightarrow (C', \delta', \varepsilon')$$

is a morphism $f\colon C \longrightarrow C'$ in $\mathcal{C}$ such that it is *comultiplicative* and *counital*, that is, the following diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\delta"'] & C \arrow[d, "\delta'"] \\ C\otimes C \arrow[r, "f\otimes f"'] & C'\otimes C' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & C' \arrow[dl, "\varepsilon'"] \\ & I & \end{tikzcd}$$

are commutative, respectively.

Exactly as we did for monoid objects and monoid morphisms, we write here the notation for the category of comonoid objects.

Definition 1.2.7. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. The *category of comonoids in* $\mathcal{C}$, denoted by $\mathbf{Comon}(\mathcal{C})$, is the category consisting of comonoid objects in $\mathcal{C}$ and comonoid morphisms between them, that is, $\mathbf{Comon}(\mathcal{C}) = (\mathbf{Mon}(\mathcal{C}^{op}))^{op}$. If $\mathcal{C}$ is a symmetric monoidal category $(\mathcal{C}, \otimes, I, B)$, we denote by $\mathbf{Comon_{coc}}(\mathcal{C})$ the full subcategory of $\mathbf{Comon}(\mathcal{C})$ consisting of cocommutative comonoids.

Now that we know both the categories of monoid and comonoid objects, we can equip them with natural monoidal structures that will allow us to talk about richer constructions afterwards. Given a symmetric monoidal category $(\mathcal{C}, \otimes, I, B)$, we define a natural monoidal structure on $\mathbf{Mon}(\mathcal{C})$ as follows: given (M_1, μ_1, η_1) and $(M_2, \mu_2, \eta_2) \in \mathbf{Mon}(\mathcal{C})$, we define

$$(M_1, \mu_1, \eta_1) \otimes (M_2, \mu_2, \eta_2) := (M_1 \otimes M_2, \mu, \eta),$$

where the product is

$$\mu = (\mu_1 \otimes \mu_2) \circ (\mathrm{id}_{M_1} \otimes B_{M_2,M_1} \otimes \mathrm{id}_{M_2}),$$

with

$$(M_1 \otimes M_2) \otimes (M_1 \otimes M_2) \xrightarrow{\mathrm{id}_{M_1} \otimes B_{M_2,M_1} \otimes \mathrm{id}_{M_2}} (M_1 \otimes M_1) \otimes (M_2 \otimes M_2)$$

and

$$(M_1 \otimes M_1) \otimes (M_2 \otimes M_2) \xrightarrow{\mu_1 \otimes \mu_2} M_1 \otimes M_2,$$

and the unit is

$$\eta = I \cong I \otimes I \xrightarrow{\eta_1 \otimes \eta_2} M_1 \otimes M_2.$$

Analogously, given $(C_1, \delta_1, \varepsilon_1)$ and $(C_2, \delta_2, \varepsilon_2) \in \mathbf{Comon}(\mathcal{C})$, we define

$$(C_1, \delta_1, \varepsilon_1) \otimes (C_2, \delta_2, \varepsilon_2) := (C_1 \otimes C_2, \delta, \varepsilon),$$

where the coproduct is given by

$$\delta = (\mathrm{id}_{M_1} \otimes B_{M_2,M_1} \otimes \mathrm{id}_{M_2}) \circ (\delta_1 \otimes \delta_2),$$

with

$$C_1 \otimes C_2 \xrightarrow{\delta_1 \otimes \delta_2} (C_1 \otimes C_1) \otimes (C_2 \otimes C_2)$$

and

$$(C_1 \otimes C_1) \otimes (C_2 \otimes C_2) \xrightarrow{\mathrm{id}_{C_1} \otimes B_{C_1,C_2} \otimes \mathrm{id}_{C_2}} (C_1 \otimes C_2) \otimes (C_1 \otimes C_2),$$

and the counit is

$$\varepsilon = C_1 \otimes C_2 \xrightarrow{\varepsilon_1 \otimes \varepsilon_2} I \otimes I \cong I.$$

With these constructions, $\mathbf{Mon}(\mathcal{C})$ and $\mathbf{Comon}(\mathcal{C})$ become symmetric monoidal categories as well and, obviously, the same applies to $\mathbf{Mon_c}(\mathcal{C})$ and $\mathbf{Comon_{coc}}(\mathcal{C})$. Having this in mind, we can finally add compatibility conditions in monoid and comonoid objects that will allow us to generalize the construction of bialgebras over a commutative ring in general symmetric monoidal categories.

Proposition 1.2.8. *Let $(\mathcal{C}, \otimes, I, B)$ be a symmetric monoidal category. Consider the following data:*

- $C \in \mathcal{C}$,
- $\mu\colon C \otimes C \longrightarrow C$, $\eta\colon I \longrightarrow C \in \mathrm{Hom}(\mathcal{C})$,
- $\delta\colon C \longrightarrow C \otimes C$, $\varepsilon\colon C \longrightarrow I \in \mathrm{Hom}(\mathcal{C})$.

Suppose that $(C, \mu, \eta) \in \mathbf{Mon}(\mathcal{C})$ and $(C, \delta, \varepsilon) \in \mathbf{Comon}(\mathcal{C})$. Then, the following conditions are equivalent:

1. *$\mu\colon (C, \delta, \varepsilon) \otimes (C, \delta, \varepsilon) \longrightarrow (C, \delta, \varepsilon)$ and $\eta\colon I \longrightarrow (C, \delta, \varepsilon)$ are comonoid morphisms.*
2. *$\delta\colon (C, \mu, \eta) \longrightarrow (C, \mu, \eta) \otimes (C, \mu, \eta)$ and $\varepsilon\colon (C, \mu, \eta) \longrightarrow I$ are monoid morphisms.*

That is, the diagrams

$$\begin{tikzcd}
C \otimes C \arrow[r, "\mu"] \arrow[d, "\delta\otimes\delta"'] & C \arrow[r, "\delta"] & C \otimes C \\
C \otimes C \otimes C \otimes C \arrow[rr, "\mathrm{id}_C \otimes B_{C,C} \otimes \mathrm{id}_C"'] & & C \otimes C \otimes C \otimes C \arrow[u, "\mu\otimes\mu"']
\end{tikzcd}$$

$$\begin{tikzcd}
C \otimes C \arrow[r, "\mu"] \arrow[d, "\varepsilon\otimes\varepsilon"'] & C \arrow[d, "\varepsilon"] \\
I \otimes I \arrow[r, "\cong"'] & I
\end{tikzcd}
\quad
\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[d, "\cong"'] & C \arrow[d, "\delta"] \\
I \otimes I \arrow[r, "\eta\otimes\eta"'] & C \otimes C
\end{tikzcd}
\quad
\begin{tikzcd}
I \arrow[rr, "\eta"] \arrow[dr, equal] & & C \arrow[dl, "\varepsilon"] \\
& I &
\end{tikzcd}$$

are commutative if and only if the following diagrams are

$$\begin{tikzcd}
C \arrow[r, "\delta"] & C \otimes C & C \otimes C \otimes C \otimes C \arrow[l, "\mu\otimes\mu"'] \\
C \otimes C \arrow[u, "\mu"] \arrow[rr, "\delta\otimes\delta"'] & & C \otimes C \otimes C \otimes C \arrow[u, "\mathrm{id}_C \otimes B_{C,C} \otimes \mathrm{id}_C"']
\end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\delta"] & C \otimes C \\ I \arrow[u, "\eta"] \arrow[r, "\cong"'] & I \otimes I \arrow[u, "\eta\otimes\eta"'] \end{tikzcd} \quad \begin{tikzcd} C \arrow[r, "\varepsilon"] & I \\ C \otimes C \arrow[u, "\mu"] \arrow[r, "\varepsilon\otimes\varepsilon"'] & I \otimes I \arrow[u, "\cong"'] \end{tikzcd} \quad \begin{tikzcd} C \arrow[rr, "\varepsilon"] & & I \\ & I. \arrow[ul, "\eta"] \arrow[ur, equal] & \end{tikzcd}$$

Definition 1.2.9. Let $(\mathcal{C}, \otimes, I, B)$ be a symmetric monoidal category. A *bimonoid object* is a quintuple $(C, \mu, \eta, \delta, \varepsilon)$ verifying the above equivalent conditions. They form a category, denoted by $\mathbf{Bimon}(\mathcal{C})$, when considering as morphisms

$$f\colon (C, \mu, \eta, \delta, \varepsilon) \longrightarrow (C', \mu', \eta', \delta', \varepsilon')$$

those morphisms $f\colon C \longrightarrow C'$ in $\mathcal{C}$ which are both monoid and comonoid morphisms. In other words,

$$\mathbf{Mon}(\mathbf{Comon}(\mathcal{C})) = \mathbf{Bimon}(\mathcal{C}) = \mathbf{Comon}(\mathbf{Mon}(\mathcal{C})).$$

We also say that a bimonoid is commutative or cocommutative whenever its underlying monoid is commutative or its underlying comonoid is cocomutative, respectively. As usual, we denote by $\mathbf{Bimon_c}(\mathcal{C})$ and $\mathbf{Bimon_{coc}}(\mathcal{C})$ the full subcategories of $\mathbf{Bimon}(\mathcal{C})$ consisting of commutative and cocommutative bimonoids, respectively. In other words,

$$\mathbf{Bimon_c}(\mathcal{C}) = \mathbf{Comon}(\mathbf{Mon_c}(\mathcal{C})) \quad \text{and} \quad \mathbf{Bimon_{coc}}(\mathcal{C}) = \mathbf{Mon}(\mathbf{Comon_{coc}}(\mathcal{C})).$$

It turns out that, if $\mathcal{C}$ is a cartesian monoidal category, a bimonoid in $\mathcal{C}$ is just a monoid object (M, μ, η) where the comonoid structure is uniquely determined: the comultiplication is the diagonal map $\Delta\colon M \longrightarrow M \times M$ and the counit is just the unique map $t\colon M \longrightarrow I$ from M to the terminal object I in $\mathcal{C}$. Therefore a group G, thought as a monoid in **Set** in which every element $x \in G$ has an inverse $S(x) \in G$, is a bimonoid in **Set** such that the following diagram is commutative

$$\begin{tikzcd} G \otimes G \arrow[rr, "S\otimes \mathrm{id}_G"] & & G \otimes G \arrow[d, "\mu"] \\ G \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "t"] & I \arrow[r, "\eta"] & G \\ G \otimes G \arrow[rr, "\mathrm{id}_G \otimes S"'] & & G \otimes G. \arrow[u, "\mu"'] \end{tikzcd}$$

Such commutativity is just asserting that, for every $x \in G$,

$$\begin{cases} S(x) * x = e, \\ x * S(x) = e, \end{cases}$$

where $\mu(-,-) = - * -$ and $e \in G$ is the unique element in the image of η. Monoids of this kind in an arbitrary symmetric monoidal category, not necessarily cartesian, are called Hopf monoids.

Definition 1.2.10. Let $(\mathcal{C}, \otimes, I, B)$ be a symmetric monoidal category. A *Hopf monoid* in $\mathcal{C}$ is a bimonoid $(H, \mu, \eta, \delta, \varepsilon)$ in $\mathcal{C}$ together with a morphism $S\colon H \longrightarrow H$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd}
H \otimes H \arrow[rr, "S\otimes \mathrm{id}_H"] & & H \otimes H \arrow[d, "\mu"] \\
H \arrow[u, "\delta"] \arrow[d, "\delta"'] \arrow[r, "\varepsilon"] & I \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "\mathrm{id}_H \otimes S"'] & & H \otimes H. \arrow[u, "\mu"']
\end{tikzcd}$$

The morphism S is called the *antipode* of the Hopf monoid. A *morphism of Hopf monoids*

$$f\colon (H, \mu, \eta, \delta, \varepsilon, S) \longrightarrow (H', \mu', \eta', \delta', \varepsilon', S')$$

is a bimonoid morphism $f\colon H \longrightarrow H'$ such that the diagram

$$\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, "S"'] & H' \arrow[d, "S'"] \\
H \arrow[r, "f"'] & H'
\end{tikzcd}$$

commutes. This defines the category $\mathbf{Hopf}(\mathcal{C})$ of Hopf monoids and Hopf monoid morphisms.

1.2.2 Monoidal Functors

Once we have defined the notion of monoidal categories, it is natural to define functors between them that preserve the respective tensor structures. For the sake of completeness we will start off with a relaxed version, though we will not need it throughout the work. As we said, our main reference here is [AM10].

Let $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$ be a pair of monoidal categories and let

$$F\colon (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}) \longrightarrow (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$$

be a functor between them. We write

$$F^2 := \otimes_{\mathcal{D}} \circ (F \times F) \quad \text{and} \quad F_2 := F \circ \otimes_{\mathcal{C}},$$

which are functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{D}$. In order not to over-saturate the diagrams, we will omit the labels of the tensor products, units and elements of the monoidal structures in general when we consider it necessary.

Definition 1.2.11. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is termed *lax monoidal* if there is a natural transformation

$$F(A) \otimes_{\mathcal{D}} F(B) \xrightarrow{\varphi_{A,B}} F(A \otimes_{\mathcal{C}} B)$$

from the functor F^2 to the functor F_2 and a morphism

$$I_{\mathcal{C}} \xrightarrow{\varphi_0} F(I_{\mathcal{D}})$$

in $\mathcal{D}$ such that the following diagrams are commutative.

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[r, "a"] \arrow[d, "\varphi_{A,B} \otimes \mathrm{id}_{F(C)}"'] & F(A) \otimes (F(B) \otimes F(C)) \arrow[d, "\mathrm{id}_{F(A)} \otimes \varphi_{A,B}"] \\
F(A \otimes B) \otimes F(C) \arrow[d, "\varphi_{A \otimes B, C}"'] & F(A) \otimes F(B \otimes C) \arrow[d, "\varphi_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F(a)"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes F(A) \arrow[r, "\lambda_{F(A)}"] \arrow[d, "\varphi_0 \otimes \mathrm{id}_{F(A)}"'] & F(A) \\
F(I) \otimes F(A) \arrow[r, "\varphi_{I,A}"'] & F(I \otimes A) \arrow[u, "F(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \otimes I \arrow[r, "\rho_{F(A)}"] \arrow[d, "\mathrm{id}_{F(A)} \otimes \varphi_0"'] & F(A) \\
F(A) \otimes F(I) \arrow[r, "\varphi_{A,I}"'] & F(A \otimes I) \arrow[u, "F(\rho_A)"']
\end{tikzcd}$$

Definition 1.2.12. We say that a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is *colax monoidal* if there is a natural transformation

$$F(A \otimes_{\mathcal{C}} B) \xrightarrow{\psi_{A,B}} F(A) \otimes_{\mathcal{D}} F(B)$$

from the functor F_2 to the functor F^2 and a morphism

$$F(I_{\mathcal{D}}) \xrightarrow{\psi_0} I_{\mathcal{C}}$$

in $\mathcal{D}$ verifying the dual axioms of definition 1.2.11.

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[r, "a"] & F(A) \otimes (F(B) \otimes F(C)) \\
F(A \otimes B) \otimes F(C) \arrow[u, "\psi_{A,B} \otimes \mathrm{id}_{F(C)}"] & F(A) \otimes F(B \otimes C) \arrow[u, "\mathrm{id}_{F(A)} \otimes \psi_{A,B}"'] \\
F((A \otimes B) \otimes C) \arrow[r, "F(a)"'] \arrow[u, "\psi_{A \otimes B, C}"] & F(A \otimes (B \otimes C)) \arrow[u, "\psi_{A, B \otimes C}"']
\end{tikzcd}$$

$$
\begin{tikzcd}
I \otimes F(A) \arrow[r, "\lambda_{F(A)}"] & F(A) \\
F(I) \otimes F(A) \arrow[u, "\psi_0 \otimes \mathrm{id}_{F(A)}"] & F(I \otimes A) \arrow[l, "\psi_{I,A}"] \arrow[u, "F(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \otimes I \arrow[r, "\rho_{F(A)}"] & F(A) \\
F(A) \otimes F(I) \arrow[u, "\mathrm{id}_{F(A)} \otimes \psi_0"] & F(A \otimes I) \arrow[l, "\psi_{A,I}"] \arrow[u, "F(\rho_A)"']
\end{tikzcd}
$$

Definition 1.2.13. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, B_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, B_{\mathcal{D}})$ be two symmetric monoidal categories. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is named *bilax monoidal* if there are natural transformations φ and ψ

$$
\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, bend left, "\varphi_{A,B}"] & F(A \otimes B) \arrow[l, bend left, "\psi_{A,B}"]
\end{tikzcd}
$$

between the functors F^2 and F_2, and morphisms

$$\varphi_0\colon I \longrightarrow F(I) \quad \text{and} \quad \psi_0\colon F(I) \longrightarrow I$$

in $\mathcal{D}$ such that (F, φ) is lax, (F, ψ) is colax and the diagrams below are commutative.

$$
\begin{tikzcd}
 & F(A \otimes B) \otimes F(C \otimes D) \arrow[dl, "\varphi_{A\otimes B, C\otimes D}"'] \arrow[dr, "\psi_{A,B} \otimes \psi_{C,D}"] & \\
F(A \otimes B \otimes C \otimes D) \arrow[d, "F(\mathrm{id}_A \otimes B_{B,C} \otimes \mathrm{id}_D)"'] & & F(A) \otimes F(B) \otimes F(C) \otimes F(D) \arrow[d, "\mathrm{id}_{F(A)} \otimes B_{F(B),F(C)} \otimes \mathrm{id}_{F(D)}"] \\
F(A \otimes C \otimes B \otimes D) \arrow[dr, "\psi_{A\otimes C, B\otimes D}"'] & & F(A) \otimes F(C) \otimes F(B) \otimes F(D) \arrow[dl, "\varphi_{A,C} \otimes \varphi_{B,D}"] \\
 & F(A \otimes C) \otimes F(B \otimes D) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \arrow[r, "\varphi_0"] \arrow[d, "\lambda_I^{-1}"'] & F(I) \arrow[r, "F(\lambda_I^{-1})"] & F(I \otimes I) \arrow[d, "\psi_{I,I}"] \\
I \otimes I \arrow[rr, "\varphi_0 \otimes \varphi_0"'] & & F(I) \otimes F(I)
\end{tikzcd}
\qquad
\begin{tikzcd}
I & F(I) \arrow[l, "\psi_0"'] & F(I \otimes I) \arrow[l, "F(\lambda_I)"'] \\
I \otimes I \arrow[u, "\lambda_I"] & & F(I) \otimes F(I) \arrow[ll, "\psi_0 \otimes \psi_0"] \arrow[u, "\varphi_{I,I}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \arrow[rr, equal] \arrow[dr, "\varphi_0"'] & & I \\
 & F(I) \arrow[ur, "\psi_0"'] &
\end{tikzcd}
$$

We will sometimes omit the word "monoidal", and say just lax, colax and bilax. Let F be a lax functor with structure maps φ and φ_0 as in definition 1.2.11. Then, we will denote such functor by (F, φ, φ_0), or (F, φ), or simply F, whenever the structure maps are understood. This convention applies analogously to colax and bilax functors.

Obviously we would like the natural transformations involved in the definitions to be in fact natural isomorphisms. This is the definition that we will make use of latter in our work.

Definition 1.2.14. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between monoidal categories and let φ and ψ be natural transformations as in 1.2.11 and 1.2.12, and let φ_0 and ψ_0 be morphisms in $\mathcal{D}$ as in 1.2.11 and 1.2.12. We say that (F, φ) is *strong* if it is lax and φ and φ_0 are invertible. We say that (F, ψ) is *costrong* if it is colax and ψ and ψ_0 are invertible. We say that (F, φ, ψ) is *bistrong* if it is bilax, strong and costrong. Moreover, we call (F, φ) or (F, ψ) *strict* if φ and φ_0 are identities, or if ψ and ψ_0 are identities, respectively, and we also call (F, φ, ψ) *strict* if the underlying lax and colax structures are.

Observe that (F, φ) is strong if and only if (F, φ^{-1}) is costrong, and it can be proven that if (F, φ, ψ) is bistrong, then $\varphi = \psi^{-1}$ (see [AM10], proposition 3.45). We now give the definitions corresponding to the symmetric case.

Definition 1.2.15. A lax (resp. colax) monoidal functor (F, φ) between two symmetric monoidal categories (resp. (F, ψ)) is *braided* if the right-hand (resp. left-hand) square is commutative

$$\begin{tikzcd}
F(A \otimes B) \arrow[r, "\psi_{A,B}"] \arrow[d, "F(B)"'] & F(A) \otimes F(B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "B"] & F(A \otimes B) \arrow[d, "F(B)"] \\
F(B \otimes A) \arrow[r, "\psi_{B,A}"'] & F(B) \otimes F(A) \arrow[r, "\varphi_{B,A}"'] & F(B \otimes A).
\end{tikzcd}$$

Analogously, a bilax monoidal functor (F, φ, ψ) is *braided* if (F, φ) and (F, ψ) are braided lax and colax monoidal functors, respectively.

As one can imagine, one of the main reasons why lax, colax and bilax monoidal functors are defined is that they preserve monoid, comonoid and bimonoid objects, respectively, in a certain sense that we will make clear below.

Proposition 1.2.16. *Let $(F, \varphi, \varphi_0)\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a lax monoidal functor. Let (M, μ, η) be a monoid object in $\mathcal{C}$. Consider the triple*

$$(F(A), F(\mu) \circ \varphi_{A,A}, F(\eta) \circ \varphi_0),$$

where $F(\mu) \circ \varphi_{A,A}$ and $F(\eta) \circ \varphi_0$ are given by the following compositions

$$F(A) \otimes F(A) \xrightarrow{\varphi_{A,A}} F(A \otimes A) \xrightarrow{F(\mu)} F(A),$$

$$I \xrightarrow{\varphi_0} F(I) \xrightarrow{F(\eta)} F(A).$$

Then, $(F(A), F(\mu) \circ \varphi_{A,A}, F(\eta) \circ \varphi_0)$ *is a monoid object in* $\mathcal{D}$*. Similarly, let*

$$(F, \psi, \psi_0) \colon \mathcal{C} \longrightarrow \mathcal{D}$$

be a colax monoidal functor and let (C, δ, ε) *be a comonoid object in* $\mathcal{C}$*. Then, the triple*

$$(F(C), \psi_{C,C} \circ F(\delta), \psi_0 \circ F(\varepsilon)),$$

where the structure maps are given by

$$F(C) \xrightarrow{F(\delta)} F(C \otimes C) \xrightarrow{\psi_{C,C}} F(C) \otimes F(C)$$

and

$$F(C) \xrightarrow{F(\varepsilon)} F(I) \xrightarrow{\psi_0} I,$$

is a comonoid object in $\mathcal{D}$*. Consider now a bilax monoidal functor*

$$(F, \varphi, \varphi_0, \psi, \psi_0) \colon \mathcal{C} \longrightarrow \mathcal{D}$$

and let $(H, \mu, \eta, \delta, \varepsilon)$ *be a bimonoid object in* $\mathcal{C}$*. Then, the quintuple*

$$(F(H), F(\mu) \circ \varphi_{A,A}, F(\eta) \circ \varphi_0, \psi_{C,C} \circ F(\delta), \psi_0 \circ F(\varepsilon))$$

is a bimonoid object in $\mathcal{D}$*. Furthermore, if*

$$f \colon H \longrightarrow H'$$

is a morphism of monoid, comonoid or bimonoid objects in $\mathcal{C}$*, then the induced morphism*

$$F(f) \colon F(H) \longrightarrow F(H')$$

is a morphism of monoid, comonoid or bimonoid objects in $\mathcal{D}$*, respectively.*

The above proposition implies that lax, colax and bilax functors from $\mathcal{C}$ to $\mathcal{D}$ induce functors

$$\mathbf{Mon}(\mathcal{C}) \longrightarrow \mathbf{Mon}(\mathcal{D}), \quad \mathbf{Comon}(\mathcal{C}) \longrightarrow \mathbf{Comon}(\mathcal{D}), \quad \mathbf{Bimon}(\mathcal{C}) \longrightarrow \mathbf{Bimon}(\mathcal{D}),$$

respectively. There is a similar result for commutative monoids and cocommutative comonoids, which states that braided (co)lax monoidal functors preserve (co)commutative (co)monoid objects, which means that braided lax and braided colax monoidal functors induce functors

$$\mathbf{Mon_c}(\mathcal{C}) \longrightarrow \mathbf{Mon_c}(\mathcal{D}), \quad \mathbf{Comon_{coc}}(\mathcal{C}) \longrightarrow \mathbf{Comon_{coc}}(\mathcal{D}),$$

respectively, which implies that braided bilax monoidal functors induce functors

$$\mathbf{Bimon_c}(\mathcal{C}) \longrightarrow \mathbf{Bimon_c}(\mathcal{D}), \quad \mathbf{Bimon_{coc}}(\mathcal{C}) \longrightarrow \mathbf{Bimon_{coc}}(\mathcal{D}).$$

Once we know a little of monoidal functors in general, we focus on the strong case, which will be of our interest later on. We recall that a strong monoidal functor is a lax monoidal functor (F, φ, φ_0) for which the structure maps are invertible. This implies (in fact, it is equivalent) that $(F, \varphi^{-1}, \varphi_0^{-1})$ is a costrong monoidal functor. Thus, it is very natural to wonder if $(F, \varphi, \varphi^{-1})$ may be bistrong. It turns out that this is the case if and only if the functor (F, φ) is braided, which is a natural thing to expect. We state the precise result below, whose proof can be found in [AM10], proposition 3.46.

Proposition 1.2.17. *Let $(F, \varphi, \psi): \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between symmetric monoidal categories. Then, the following are equivalent:*

1. *(F, φ, ψ) is bistrong.*
2. *(F, φ) is braided strong, $\psi = \varphi^{-1}$ and $\psi_0 = \varphi_0^{-1}$.*
3. *(F, ψ) is braided costrong, $\psi = \varphi^{-1}$ and $\psi_0 = \varphi_0^{-1}$.*

1.2.3 Bicategories

There are several ways to introduce the notion of bicategory. One possible way, keeping in mind what we have done so far, is the following: as we have said, a classical monoid in **Set** may be regarded as a one-object category. Hence, a category can be thought as a multi-object version of a monoid. In a similar fashion, a bicategory is, roughly speaking, a multi-object version of a monoidal category.

Definition 1.2.18. A *bicategory* $\mathcal{B}$ consists of the following data:

Objects. A class of objects $\mathrm{Obj}(\mathcal{B}) = \mathcal{B}_0$, whose elements are called *objects* or 0-*cells* in $\mathcal{B}$. If $X \in \mathcal{B}_0$, we usually write $X \in \mathcal{B}$.

Hom categories. For each pair of objects $X, Y \in \mathcal{B}$, there is a category $\mathcal{B}(X, Y)$ called *hom category*.

- Its objects are called 1-*cells* in $\mathcal{B}$, and the collection of all 1-cells from all hom categories is denoted by $\mathcal{B}_1$.
- Its arrows are called 2-*cells* in $\mathcal{B}$. The collection of all 2-cells from all hom categories is denoted by $\mathcal{B}_2$.
- Composition and identity arrows in the hom category $\mathcal{B}(X, Y)$ are named *vertical composition* and *identity* 2-*cells*, respectively.
- Isomorphisms in $\mathcal{B}(X, Y)$ are called *invertible* 2-*cells*, and their inverses are named *vertical inverses*.
- As usual, for every 1-cell, id_f stands for its identity 2-cell.

As in ordinary category theory, we require the hom categories $\mathcal{B}(X, Y)$ to be disjoint. If that is not the case, we just replace them with their disjoint union. We recall that, since each hom category $\mathcal{B}(X, Y)$ is an ordinary category, in particular we have that the vertical composition of 2-cells is associative and unital in the strict sense.

This means that, for 1-cells f, f', f'' and $f''' \in \mathcal{B}(X,Y)$; and 2-cells $\alpha\colon f \longrightarrow f'$, $\alpha'\colon f' \longrightarrow f''$ and $\alpha''\colon f'' \longrightarrow f'''$, the equalities

$$\begin{cases} (\alpha'' \circ \alpha') \circ \alpha = \alpha'' \circ (\alpha' \circ \alpha), \\ \alpha = \alpha \circ \mathrm{id}_f = \mathrm{id}_{f'} \circ \alpha \end{cases}$$

hold in $\mathcal{B}(X,Y)(f,f''')$ and $\mathcal{B}(X,Y)(f,f')$, respectively. For 1-cells $f, f' \in \mathcal{B}(X,Y)$, we may represent each 2-cell $\alpha\colon f \longrightarrow f'$ using the diagram

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "f'"'{name=D}] & Y. \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

Identity 1-cells. For each object $X \in \mathcal{B}$, there is an associated 1-cell $\mathrm{id}_X \in \mathcal{B}(X,X)$ called the *identity* 1-*cell of* X.

Horizontal composition. For each triple of objects $X, Y, Z \in \mathcal{B}$, there is a functor

$$C_{X,Y,Z}\colon \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \longrightarrow \mathcal{B}(X,Z)$$

called the *horizontal composition.* For each pair of 1-cells $(f,g) \in \mathcal{B}(Y,Z)\times\mathcal{B}(X,Y)$ and each pair of 2-cells $(\alpha,\beta) \in \mathcal{B}(Y,Z)\times\mathcal{B}(X,Y)$, we write

$$\begin{cases} C_{X,Y,Z}(f,g) = f \circ g, \\ C_{X,Y,Z}(\alpha,\beta) = \alpha * \beta. \end{cases}$$

Using the fancy notation we have given above for the 2-cells, the horizontal composition $C_{X,Y,Z}$ is the assignment given by

$$\begin{tikzcd} X \arrow[r, bend left=50, "g"{name=U1}] \arrow[r, bend right=50, "g'"'{name=D1}] & Y \arrow[r, bend left=50, "f"{name=U2}] \arrow[r, bend right=50, "f'"'{name=D2}] & Z \arrow[r, mapsto] & X \arrow[r, bend left=50, "f\circ g"{name=U3}] \arrow[r, bend right=50, "f'\circ g'"'{name=D3}] & Z. \arrow[from=U1, to=D1, Rightarrow, "\beta"] \arrow[from=U2, to=D2, Rightarrow, "\alpha"] \arrow[from=U3, to=D3, Rightarrow, "\alpha*\beta"] \end{tikzcd}$$

The functoriality of the horizontal composition $C_{X,Y,Z}$ means the following:

- It preserves identity 2-cells, that is, the equality

  $$\mathrm{id}_f * \mathrm{id}_g = \mathrm{id}_{f\circ g}$$

  holds in $\mathcal{B}(X,Z)(f\circ g, f\circ g)$ for every $g \in \mathcal{B}(X,Y)$ and every $f \in \mathcal{B}(Y,Z)$. This may be visualized as

$$\begin{tikzcd} X \arrow[r, bend left=50, "g"{name=U1}] \arrow[r, bend right=50, "g"'{name=D1}] & Y \arrow[r, bend left=50, "f"{name=U2}] \arrow[r, bend right=50, "f"'{name=D2}] & Z \arrow[r, mapsto] & X \arrow[r, bend left=50, "f\circ g"{name=U3}] \arrow[r, bend right=50, "f\circ g"'{name=D3}] & Z \arrow[r, phantom, "="] & X \arrow[r, bend left=50, "f\circ g"{name=U4}] \arrow[r, bend right=50, "f\circ g"'{name=D4}] & Z. \arrow[from=U1, to=D1, Rightarrow, "\mathrm{id}_g"] \arrow[from=U2, to=D2, Rightarrow, "\mathrm{id}_f"] \arrow[from=U3, to=D3, Rightarrow, "\mathrm{id}_f * \mathrm{id}_g"] \arrow[from=U4, to=D4, Rightarrow, "\mathrm{id}_{f\circ g}"] \end{tikzcd}$$

- It preserves vertical composition, that is, the equality

$$(\alpha' \circ \alpha) * (\beta' \circ \beta) = (\alpha' * \beta') \circ (\alpha * \beta)$$

holds in $\mathcal{B}(X,Z)(f\circ g, f''\circ g'')$. This equality is called the *middle four exchange*, and it may be visualized as the equality of the two ways to compose the diagram

$$\begin{tikzcd} X \arrow[r, bend left=50, "g"] \arrow[r, "g'"] \arrow[r, bend right=50, "g''"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & Y \arrow[r, bend left=50, "f"] \arrow[r, "f'"] \arrow[r, bend right=50, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & Z \end{tikzcd}$$

so as to get a 2-cell.

Associator. For every $W, X, Y, Z \in \mathcal{B}$, there is a natural isomorphism

$$a_{W,X,Y,Z} \colon C_{W,X,Z} \circ (C_{X,Y,Z} \times \mathrm{id}_{\mathcal{B}(W,X)}) \longrightarrow C_{W,Y,Z} \circ (\mathrm{id}_{\mathcal{B}(Y,Z)} \times C_{W,X,Y})$$

called the *associator*, where the corresponding functors go from $\mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \times \mathcal{B}(W,X)$ to $\mathcal{B}(W,Z)$, identifying

$$\big(\mathcal{B}(Y,Z) \times \mathcal{B}(X,Y)\big) \times \mathcal{B}(W,X) = \mathcal{B}(Y,Z) \times \big(\mathcal{B}(X,Y) \times \mathcal{B}(W,X)\big).$$

This essentially means that horizontal composition is associative up to the specified natural isomorphism given by a, that is, for 1-cells $f \in \mathcal{B}(W,X)$, $g \in \mathcal{B}(X,Y)$ and $h \in \mathcal{B}(Y,Z)$, there is an invertible 2-cell

$$a_{h,g,f} \colon (h \circ g) \circ f \xrightarrow{\ \cong\ } h \circ (g \circ f)$$

in $\mathcal{B}(W,Z)$. The naturality of a means that, for 2-cells $\alpha \colon f \longrightarrow f'$, $\beta \colon g \longrightarrow g'$ and $\gamma \colon h \longrightarrow h'$

$$\begin{tikzcd} W \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & X \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & Y \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "h'"'] \arrow[r, phantom, "\Downarrow\gamma"] & Z, \end{tikzcd}$$

the diagram in $\mathcal{B}(W,Z)$

$$\begin{tikzcd} (h \circ g) \circ f \arrow[r, "a_{h,g,f}"] \arrow[d, "(\gamma*\beta)*\alpha"'] & h \circ (g \circ f) \arrow[d, "\gamma*(\beta*\alpha)"] \\ (h' \circ g') \circ f' \arrow[r, "a_{h',g',f'}"'] & h' \circ (g' \circ f') \end{tikzcd}$$

is commutative.

Left and right unitors. For every pair of objects $X, Y \in \mathcal{B}$, there are two natural isomorphisms

$$C_{X,Y,Y} \circ (\mathrm{id}_Y \times \mathrm{id}_{\mathcal{B}(X,Y)}) \xrightarrow{l_{X,Y}} \mathrm{id}_{\mathcal{B}(X,Y)} \xleftarrow{r_{X,Y}} C_{X,X,Y} \circ (\mathrm{id}_{\mathcal{B}(X,Y)} \times \mathrm{id}_X)$$

called the *left unitor* and the *right unitor*, respectively. As before, this just means that horizontal composition is unital up to the specified natural isomorphisms given by l and r. We will often omit the subscripts in all these natural isomorphisms.

Finally, for 1-cells $f \in \mathcal{B}(V,W)$, $g \in \mathcal{B}(W,X)$, $h \in \mathcal{B}(X,Y)$ and $k \in \mathcal{B}(Y,Z)$, the above data is required to verify the following pair of axioms.

Unit axiom. The following diagram in $\mathcal{B}(V,X)$

$$\begin{tikzcd}
(g \circ \mathrm{id}_W) \circ f \arrow[rr, "a"] \arrow[dr, "r_g * \mathrm{id}_f"'] & & g \circ (\mathrm{id}_W \circ f) \arrow[dl, "\mathrm{id}_g * l_f"] \\
& g \circ f &
\end{tikzcd}$$

is commutative.

Pentagon axiom. The diagram

$$\begin{tikzcd}
& (k \circ h) \circ (g \circ f) \arrow[dr, "a_{k,h,g\circ f}"] & \\
((k \circ h) \circ g) \circ f \arrow[ur, "a_{k\circ h,g,f}"] \arrow[d, "a_{k,h,g} * \mathrm{id}_f"'] & & k \circ (h \circ (g \circ f)) \\
(k \circ (h \circ g)) \circ f \arrow[rr, "a_{k,h\circ g,f}"'] & & k \circ ((h \circ g) \circ f). \arrow[u, "\mathrm{id}_k * a_{h,g,f}"']
\end{tikzcd}$$

in $\mathcal{B}(V,Z)$ is commutative.

As we said when we where introducing the definition, one of the most basic examples of bicategory is a monoidal category, since it may be regarded as a bicategory with one object. For us, an important example will be the bicategory of spans in some category with pullbacks, which we will explain in detal in chapter 3.

Chapter 2

Cell-Sets

2.1 The Concept of Multiset

The preliminary step before going towards the main object of study, namely cell-sets, is the concept of multiset. There are many definitions in the mathematical literature, but of course they are all based on the same intuitive idea: a multiset is a set in which we are allowed to repeat elements.

Many mathematicians have studied them at different levels of formalism. In particular Monro, in [Mon87], builds them in an appropriate way from the categorical point of view: for him multisets are just sets with some specified equivalence relation, and morphisms between multisets are maps between the underlying sets that preserve the equivalence relations. He also introduces the notion of *multinumber* which, roughly speaking, is a function from a given set to the natural numbers. Despite this notion may be more intuitive, it does not have a categorical structure as rich as the former one.

For the purpose of our work, however, we are only interested in the notion of submultiset of a given set, and the one that fits better with our intentions is the notion of *labelled submultiset*.

Definition 2.1.1. Let A be a set. A *labelled submultiset* of A is an isomorphism class in the slice category $\mathbf{Set}/A$. The terminology comes from the fact that giving a map

$$\begin{array}{rcl} a\colon I & \longrightarrow & A \\ i & \longmapsto & a_i := a(i) \end{array}$$

is the same as giving an *indexed family* $(a_i)_{i\in I}$ of elements in A indexed by I. When we talk about submultisets of A, we will usually refer to an object $a\colon I \longrightarrow A \in \mathbf{Set}/A$ instead of its isomorphism class, that is, we will drop the brackets in the expression $[a\colon I \longrightarrow A]$, though we will be very careful showing that things work well under changes of representative, unless it is really clear. Given a submultiset $a\colon I \longrightarrow A$, we usually write $a = (a_i)_{i\in I}$. Given $y \in A$, we write $y \in a = (a_i)_{i\in I}$ if there exists $i \in I$ such that $y = a_i$. We define the *cardinality* of a as the cardinality of its index set I, and we write $|a| := |I|$. This definition is obviously well-defined modulo isomorphism. We say that the submultiset a is *finite* if its cardinality is finite. From now on, we will refer to the notion of labelled submultiset only using the term *submultiset*. For every set A, we denote the set of all submultisets of A by

$$\overline{\mathbf{Set}}/A := \mathrm{Obj}(\mathbf{Set}/A)/\sim .$$

This definition is really suitable, since it allows us to repeat elements without the need of defining an equivalence relation on a set, and we can also recover the notion of multinumber just by counting the cardinality of the fibers. We can also recover the underlying set of a given submultiset by considering its image in A. Moreover, observe that this definition includes the notion of ordinary subset when we consider injective maps instead of general maps. In particular, if A is a set and $B \subseteq A$ is an ordinary subset, we can regard B as a submultiset of A via the inclusion map

$$\begin{array}{rcl} i = i_{B,A} \colon B & \hookrightarrow & A \\ b & \longmapsto & i(b) = b. \end{array}$$

Unless otherwise stated, this is the canonical representative that we will choose for a subset $B \subseteq A$, and we identify $B = i = (b)_{b \in B}$. If B consists only of a single element b, we just write $B = \{b\}$ instead of $B = (b)_{b \in \{b\}}$.

Once the correct framework for working with submultisets is established, we only need to define the basic operations with submultisets that we will use throughout the work.

Definition 2.1.2. Let $I \xrightarrow{a} A, J \xrightarrow{b} A \in \mathbf{Set}/A$. We define their *union* $a \cup b$ as the unique submultiset of A that makes the following diagram commutative

$$\begin{tikzcd}
 & J \arrow[d] \arrow[ddr, bend left, "b"] & \\
I \arrow[r] \arrow[drr, bend right, "a"'] & I \amalg J \arrow[dr, dashed, "\exists!\, a\cup b"] & \\
 & & A.
\end{tikzcd}$$

In indexed-family notation, if $a = (a_i)_{i \in I}$ and $b = (b_j)_{j \in J}$, then

$$a \cup b = (c_k)_{k \in I \amalg J},$$

where, for every $k \in I \amalg J$,

$$c_k = \begin{cases} a_k, & \text{if } k \in I, \\ b_k, & \text{if } k \in J. \end{cases}$$

We also define their *product* $a \times b$ as the unique submultiset of $A \times A$ such that the diagram

$$\begin{tikzcd}
I \arrow[d, "a"'] & I \times J \arrow[l, "p_I"'] \arrow[r, "p_J"] \arrow[dl, "a\circ p_I"'] \arrow[d, dashed, red, "\exists!\, a\times b"] \arrow[dr, "b\circ p_J"] & J \arrow[d, "b"] \\
A & A \times A \arrow[l, "p_A^1"'] \arrow[r, "p_A^2"] & A
\end{tikzcd}$$

is commutative. Equivalently, $a \times b$ is given by

$$a \times b = ((a_i, b_j))_{(i,j)\in I\times J}.$$

Of course, these definitions can be generalized to the case where we have an arbitrary family of submultisets of a given set, and the product can be in fact defined for submultisets of different sets in a completely analogous way. Before doing that, it is necessary to point out some aspects of this definition. We must check that this definition is well-defined, and this means we have to see that not only it is well-defined under a change of representative of submultisets, but also that it is well-defined under a change of the model of the disjoint union of index sets. Let us be more precise with the following proposition.

Proposition 2.1.3. *Let $I \xrightarrow{a} A, I' \xrightarrow{a'} A, J \xrightarrow{b} A, J' \xrightarrow{b'} A \in \mathbf{Set}/A$. Then, we have that*

$$a \cong a' \text{ and } b \cong b' \implies a \cup b \cong a' \cup b'.$$

Moreover, consider $I \xrightarrow{a} A, J \xrightarrow{b} A \in \mathbf{Set}/A$. Let

$$I \xrightarrow{i_1} I \amalg J \xleftarrow{i_2} J \qquad I \xrightarrow{\tilde{i_1}} \widetilde{I \amalg J} \xleftarrow{\tilde{i_2}} J$$

be two models for the disjoint union of the sets I and J. Then,

$$[I \amalg J \longrightarrow A] = [\widetilde{I \amalg J} \longrightarrow A].$$

Proof. The proof is really straightforward, but one has to be careful with the diagrams. On the one hand, we have the following diagram

$$\begin{tikzcd}
 & & I' \amalg J' \arrow[d, dashed, red, "\varphi_1 \amalg \varphi_2"'] \arrow[dd, dashed, red, bend left, "a' \cup b'"] & & \\
I' \arrow[urr, bend left, "i_1'"] \arrow[r, "\varphi_1"] \arrow[drr, bend right, "a'"'] & I \arrow[r, "i_1"] \arrow[dr, bend right, "a"'] & I \amalg J \arrow[d, dashed, red, "a\cup b"] & J \arrow[l, "i_2"'] \arrow[dl, bend left, "b"] & J' \arrow[l, "\varphi_2"'] \arrow[ull, bend right, "i_2'"'] \arrow[dll, bend left, "b'"] \\
 & & A & &
\end{tikzcd}$$

where φ_1, φ_2 are the corresponding isomorphisms. One can easily check that the red dashed diagram is commutative, which proves our claim. Indeed, we have to check that

$$(a \cup b) \circ (\varphi_1 \amalg \varphi_2) = a' \cup b',$$

so it is enough, by definition, to see that

$$(a \cup b) \circ (\varphi_1 \amalg \varphi_2) \circ i'_1 = a' \text{ and } (a \cup b) \circ (\varphi_1 \amalg \varphi_2) \circ i'_2 = b',$$

but this is trivially true. On the other hand, let us see that we have the same property when we change the disjoint union model. We have the following diagram

$$\begin{tikzcd}
 & \widetilde{I \amalg J} \arrow[d, dashed, "\varphi"'] \arrow[dd, dashed, bend left, "\widetilde{a\cup b}"] & \\
I \arrow[ur, "\tilde{i}_1"] \arrow[r, "i_1"] \arrow[dr, "a"'] & I \amalg J \arrow[d, dashed, "a\cup b"] & J \arrow[ul, "\tilde{i}_2"'] \arrow[l, "i_2"'] \arrow[dl, "b"] \\
 & A &
\end{tikzcd}$$

where φ is the bijection that exists by universality. As before, one can easily check that the red dashed diagram is commutative. ■

The reader can check that there exists an analogous statement for the product of submultisets, and therefore both union and product are well-defined. Now it is time to define the natural generalizations of union and product, for which there is an analogous statement ensuring that they are also well-defined in the same sense as we have done with the finite case.

Definition 2.1.4. Let $(a^i\colon K_i \longrightarrow A)_{i\in I}$ be a family of submultisets of a given set A. We define their *union*

$$\bigcup_{i\in I} a^i$$

as the unique submultiset of A that makes the following diagram commutative

$$\begin{tikzcd}
K_i \arrow[r] \arrow[rr, bend right, "a^i"'] & \coprod_{i\in I} K_i \arrow[r, dashed, "\exists! \cup_{i\in I} a^i"] & A
\end{tikzcd}$$

for every $i \in I$. Equivalently, if we write $a^i = (a^i_k)_{k\in K_i}$, then

$$\bigcup_{i\in I} a^i = (z_k)_{k \in \coprod_{i\in I} K_i}$$

where, for every $k \in \coprod_{i\in I} K_i$, $z_k = a^i_k$ whenever $k \in K_i$. We can also generalize their *product*

$$\bigtimes_{i\in I} a^i$$

in the following way: it corresponds to the unique submultiset of $\bigtimes_{i\in I} A$ such that the following diagram

$$\begin{tikzcd}
K_i \arrow[d, "a^i"'] & \bigtimes_{i\in I} K_i \arrow[l, "p_i"'] \arrow[dl, "a^i \circ p_i"] \arrow[d, dashed, "\exists!\, \times_{i\in I} a^i"] \\
A & \bigtimes_{i\in I} A \arrow[l, "p^i_A"]
\end{tikzcd}$$

is commutative for every $i \in I$. This is the same as saying that

$$\bigtimes_{i\in I} a^i = ((a^i_{\alpha(i)})_{i\in I})_{\alpha\in \times_{i\in I} K_i}.$$

Even though some readers could complain about the use of isomorphism classes instead of objects in the slice category, here is a simple example that shows its need. If $a = (a_i)_{i\in I} \in \mathbf{Set}/A$, we would like to have that

$$\bigcup_{i\in I} \{a_i\} = a,$$

where the union is taken in the above sense. There are infinitely many ways of regarding each $\{a_i\}$ as a submultiset, but the canonical choice is

$$\{a_i\} = \begin{array}{rccc} f_i \colon & \{a_i\} & \longrightarrow & A \\ & a_i & \longmapsto & a_i. \end{array}$$

But then

$$\bigcup_{i\in I} \{a_i\} = \coprod_{i\in I} \{a_i\} \xrightarrow{\ \bigcup_{i\in I}\{a_i\}\ } A,$$

so a cannot be equal to $\bigcup_{i\in I} \{a_i\}$, but these two objects are clearly isomorphic in $\mathbf{Set}/A$. Of course, one could always try to choose the appropriate representatives in order to have strict equality, but this is not comfortable.

2.2 The Category of Cell-Sets

Now we are ready to construct the category **Cell** of cell-sets. Whenever we talk about submultisets or any other notion related to them, we will do so in the sense of section 2.1.

Definition 2.2.1. The objects of **Cell** are triples $(\mathscr{C}, \sim, d)$ called *cell-sets*, where $\mathscr{C}$ is a set, $\sim$ is an equivalence relation on $\mathscr{C}$ and $d \colon \mathscr{C} \longrightarrow \mathbb{Z}$ is a map called *dimension function* which is constant on equivalence classes, that is, $x \sim y$ implies $d(x) = d(y)$ for every $x, y \in \mathscr{C}$. In other words, there exists a unique map $\widetilde{d}$ such that the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "d"] \arrow[d, two heads] & \mathbb{Z} \\
\mathscr{C}/\sim \arrow[r, dashed, "\widetilde{d}"] & \mathrm{Im}(d) \arrow[u, hook]
\end{tikzcd}$$

is commutative. We will also assume that $\mathrm{Im}(d) \subseteq \mathbb{N}$, that is, for us the dimensions are always non-negative integers. As usual, we just write $\mathscr{C}$ whenever the rest of the structure is understood. For every $n \in \mathbb{Z}$, we define $\mathscr{C}_n := d^{-1}(n)$ as the elements of $\mathscr{C}$ of dimension n. Then, we say that a cell-set is *pointed* if $\mathscr{C}_0$ contains a distinguished equivalence class, written as $1_{\mathscr{C}}$, whose representatives are called *base points*; and we call $\mathscr{C}$ *connected* if $\mathscr{C}_0$ consists of exactly one equivalence class.

The simplest cell-set is the *trivial* cell-set $\mathscr{T}$: it consists of a set with one element, and it is naturally connected and pointed. Another common example is the object set of a small category, with equivalence relation given by isomorphism. Since in practice many cell-sets arise in this way, such an example deserves a better name.

Definition 2.2.2. A small category $\mathcal{C}$ is termed *cell-category* if its set of objects comes equipped with a dimension function $d\colon \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathbb{Z}$ constant on isomorphism classes. The set of objects of $\mathcal{C}$ is then a cell-set called the *object cell-set of* $\mathcal{C}$.

The next natural step is to define morphisms between cell-sets. First, we need to specify an equivalence relation on the set of submultisets of a given cell-set.

Definition 2.2.3. Let $\mathscr{C}$ be a cell-set and let $I \xrightarrow{a} \mathscr{C}, J \xrightarrow{b} \mathscr{C} \in \overline{\mathbf{Set}}/\mathscr{C}$ be a pair of submultisets of $\mathscr{C}$. We write $a \sim b$ if there exists a bijection $\varphi\colon I \longrightarrow J$ such that the following diagram

$$\begin{tikzcd} I \arrow[rr, "\varphi"] \arrow[d, "a"'] & & J \arrow[d, "b"] \\ \mathscr{C} \arrow[dr] & & \mathscr{C} \arrow[dl] \\ & \mathscr{C}/\sim & \end{tikzcd}$$

is commutative. If $a = (a_i)_{i\in I}$ and $b = (b_j)_{j\in J}$, the commutativity of this diagram is the same as demanding that, for every $i \in I$,

$$a_i \sim b_{\varphi(i)} \text{ in } \mathscr{C}.$$

The above construction is obviously a well-defined equivalence relation on $\overline{\mathbf{Set}}/\mathscr{C}$.

Proposition 2.2.4. *The binary relation defined on $\overline{\mathbf{Set}}/\mathscr{C}$ is well-defined and is an equivalence relation.*

Proof. First we check that it is well-defined. Let $a, a', b, b' \in \mathbf{Set}/\mathscr{C}$ such that

$$a \cong a' \quad \text{and} \quad b \cong b'.$$

We have to see that

$$a \sim b \iff a' \sim b'.$$

Obviously, by symmetry it is enough to show only one implication, so let us assume that $a \sim b$. We have the following diagram

$$\begin{tikzcd} I' \arrow[r, "\varphi_1"] \arrow[d, "a'"] & I \arrow[rr, "\varphi"] \arrow[d, "a"'] & & J \arrow[r, "\varphi_2"] \arrow[d, "b"] & J' \arrow[d, "b'"] \\ \mathscr{C} \arrow[r, equal] & \mathscr{C} \arrow[dr] & & \mathscr{C} \arrow[dl] \arrow[r, equal] & \mathscr{C} \\ & & \mathscr{C}/\sim, & & \end{tikzcd}$$

where φ_1, φ and φ_2 are the bijections that exist by assumption. Since both squares and the inner pentagon commute by hypothesis, so does the outer pentagon, therefore the bijection $\psi := \varphi_2 \circ \varphi \circ \varphi_1$ tells us that $a' \sim b'$. The properties of reflexivity, symmetry and transitivity are trivially satisfied. ∎

Remark 2.2.5. In [RS98], the authors only define such an equivalence relation on the set of ordinary subsets of a given cell-set, and they do it in the following similar way: if $U, V \subseteq \mathscr{C}$ are subsets of the underlying set of a given cell-set $\mathscr{C}$, they are said to be equivalent, which is denoted as $U \sim V$, if there exists a bijection

$$\varphi \colon U \longrightarrow V$$

such that

$$x \sim \varphi(x)$$

for every $x \in U$. However, they do not make explicit how this definition may be generalized to the case of submultisets, nor they establish their notion of submultiset to work with. Therefore, we have chosen the definition of submultiset in 2.1 so as to give an appropriate generalization from subsets to submultisets that takes into account both the bijection, that is, both submultisets must have the same cardinality counting multiplicities, and the equivalence of the corresponding elements under such a bijection. In fact, their definition can be written as the commutativity of the diagram

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[d, "i_U"'] & & V \arrow[d, "i_V"] \\ \mathscr{C} \arrow[dr] & & \mathscr{C} \arrow[dl] \\ & \mathscr{C}/\sim, & \end{tikzcd}$$

where i_U and i_V are the corresponding inclusions.

Now that we have established the equivalence relation, we can proceed to define morphisms in **Cell**.

Definition 2.2.6. Given two cell-sets $(\mathscr{C}, \sim_{\mathscr{C}}, d_{\mathscr{C}})$ and $(\mathscr{D}, \sim_{\mathscr{D}}, d_{\mathscr{D}})$, a *cell-map*

$$f \colon (\mathscr{C}, \sim_{\mathscr{C}}, d_{\mathscr{C}}) \longrightarrow (\mathscr{D}, \sim_{\mathscr{D}}, d_{\mathscr{D}})$$

is a map

$$f \colon \mathscr{C} \longrightarrow \overline{\mathbf{Set}}/\mathscr{D}$$

defined as follows: for every $x \in \mathscr{C}$, $f(x)$ is a finite submultiset of $\mathscr{D}$ in such a way that the equivalence relations are respected, that is, if $x \sim_{\mathscr{C}} y$, then $f(x) \sim_{\mathscr{D}} f(y)$ in the sense that we have commented before. Moreover, we also ask f to preserve dimensions: for every $x \in \mathscr{C}$, every element $y \in f(x)$ has dimension $d_{\mathscr{D}}(y) = d_{\mathscr{C}}(x)$. Given two cell-maps $g\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $f\colon \mathscr{D} \longrightarrow \mathscr{E}$, their composition is defined as follows: for every $x \in \mathscr{C}$, write $g(x) = (a_i)_{i\in I}$. Then,

$$(f \circ g)(x) := \bigcup_{i \in I} f(a_i).$$

Sometimes we will also write

$$(f \circ g)(x) = \bigcup_{y \in g(x)} f(y),$$

but we want to emphasize that this means that the union is taken over all elements of the multiset $g(x)$, counting multiplicities. Obviously, the identity cell-map associated to a cell-set $\mathscr{C}$ is defined as $\mathrm{id}_{\mathscr{C}}(x) = \{x\}$, for every $x \in \mathscr{C}$.

Notation 2.2.7. Given a pair of cell-sets $\mathscr{C}$ and $\mathscr{D}$, we usually write either

$$f\colon \mathscr{C} \longrightarrow \mathscr{D} \quad \text{or} \quad f\colon \mathscr{D} \longrightarrow \overline{\mathbf{Set}}/\mathscr{D}$$

to denote a cell-map f between $\mathscr{C}$ and $\mathscr{D}$.

At this point, it may not be clear why do we need to work with submultisets in the sense of 2.1. We want to emphasize that, in practice, all the examples of cell-maps that we consider in this work can be described using only ordinary set theory. But it turns out, perhaps as a surprise, that we need to work with multisets in order to ensure the properties of composition, in particular to guarantee that the composition of two cell-maps is again a cell-map.

Proposition 2.2.8. **Cell** *is, indeed, a category.*

Proof. First we check that the composition is well-defined. As we said above, this property holds because of the use of submultisets. Let

$$\mathscr{C} \xrightarrow{g} \mathscr{D} \xrightarrow{f} \mathscr{E}$$

be a pair of composable cell-maps. Since f and g are cell-maps, it is obviously true that $f \circ g$ is dimension-preserving: for every $x \in \mathscr{C}$, write $g(x) = (a_i)_{i\in I}$ and, for every $i \in I$, write $f(a_i) = (b^i_j)_{j \in J_i}$. Then, for every $y \in \mathscr{E}$ such that

$$y \in (f \circ g)(x) = \bigcup_{i \in I} (b^i_j)_{j \in J_i},$$

there exists a pair of indices $(i, j) \in I \times J$ such that $y = b^i_j$. Since g is a cell-map, we have that $d_{\mathscr{D}}(a_i) = d_{\mathscr{C}}(x)$ and, since f is a cell-map, we have that $d_{\mathscr{E}}(y) = d_{\mathscr{E}}(b^i_j) = d_{\mathscr{D}}(a_i)$. Hence, $d_{\mathscr{E}}(y) = d_{\mathscr{C}}(x)$. The difficult part is showing that $f \circ g$ preserves the equivalence relations. Let $x, x' \in \mathscr{C}$ such that $x \sim_{\mathscr{C}} x'$. Since g is a cell-map, we have that

$$(y_i)_{i\in I} = g(x) \sim_{\mathscr{D}} g(x') = (y'_j)_{j\in J}.$$

This means that there exists a bijection $\varphi\colon I \longrightarrow J$ such that, for every $i \in I$,

$$y_i \sim y'_{\varphi(i)}.$$

Write

$$\begin{cases} f(y_i) = (z^i_k)_{k \in K_i} \\ f(y'_{\varphi(i)}) = (w^i_l)_{l \in L_i}. \end{cases}$$

Using the fact that f is a cell-map as well it follows that, for every $i \in I$, there exists a bijection $\psi_i\colon K_i \longrightarrow L_i$ such that, for every $k \in K_i$,

$$z^i_k \sim w^i_{\psi_i(k)}. \tag{2.1}$$

Using this data, we have to show that

$$(f \circ g)(x) = \bigcup_{i \in I} f(y_i) \sim \bigcup_{i \in I} f(y'_{\varphi(i)}) = (f \circ g)(x').$$

On the one hand,

$$\bigcup_{i \in I} f(y_i) = \ \coprod_{i \in I} K_i \xrightarrow{\ \bigcup_{i \in I} z^i\ } \mathscr{E}$$

where, for every $i \in I$,

$$z^i\colon K_i \longrightarrow \mathscr{E} = (z^i_k)_{k \in K_i}.$$

On the other hand,

$$\bigcup_{i \in I} f(y'_{\varphi(i)}) = \ \coprod_{i \in I} L_i \xrightarrow{\ \bigcup_{i \in I} w^i\ } \mathscr{E}$$

where, for every $i \in I$,

$$w^i\colon L_i \longrightarrow \mathscr{E} = (w^i_l)_{l \in L_i}.$$

Therefore, since the disjoint union of bijective maps is again bijective, we have that the map

$$\coprod_{i \in I} \psi_i\colon \coprod_{i \in I} K_i \longrightarrow \coprod_{i \in I} L_i$$

is a bijection that, in addition, makes the following diagram commutative

$$\begin{tikzcd}
\coprod_{i \in I} K_i \arrow[rr, "\coprod_{i \in I} \psi_i"] \arrow[d, "\bigcup_{i \in I} z^i"'] & & \coprod_{i \in I} L_i \arrow[d, "\bigcup_{i \in I} w^i"] \\
\mathscr{E} \arrow[dr] & & \mathscr{E} \arrow[dl] \\
& \mathscr{E}/\sim &
\end{tikzcd}$$

as a consequence of the fact that the following diagrams

$$\begin{tikzcd}
K_i \arrow[rr, "\psi_i"] \arrow[d, "z^i"'] & & L_i \arrow[d, "w^i"] \\
\mathscr{E} \arrow[dr] & & \mathscr{E} \arrow[dl] \\
& \mathscr{E}/\sim &
\end{tikzcd}$$

are commutative for every $i \in I$, as we have seen in 2.1.

Now that we have a well-defined composition in **Cell**, we verify the associativity property: consider a sequence of three composable cell-maps

$$\mathscr{C} \xrightarrow{f} \mathscr{D} \xrightarrow{g} \mathscr{E} \xrightarrow{h} \mathscr{F}.$$

Then, for every $x \in \mathscr{C}$,

$$((h \circ g) \circ f)(x) = \bigcup_{y \in f(x)} (h \circ g)(y) = \bigcup_{y \in f(x)} \left(\bigcup_{z \in g(y)} h(z) \right) = \bigcup_{z \in (g \circ f)(x)} h(z) = (h \circ (g \circ f))(x).$$

Finally, let us check the unit property. For every cell-map $f \colon \mathscr{C} \longrightarrow \mathscr{D}$ and every $x \in \mathscr{C}$,

$$(f \circ \mathrm{id}_{\mathscr{C}})(x) = \bigcup_{y \in \mathrm{id}_{\mathscr{C}}(x)} f(y) = \bigcup_{y \in \{x\}} f(y) = f(x),$$

and

$$(\mathrm{id}_{\mathscr{C}} \circ f)(x) = \bigcup_{y \in f(x)} \mathrm{id}_{\mathscr{C}}(y) = \bigcup_{y \in f(x)} \{y\} = f(x).$$

Therefore, **Cell** is a category, as claimed. ■

Throughout our exposition of cell-sets, specially when we consider monoid and comonoid objects in **Cell**, it is important to restrict ourselves to a certain subcategory of **Cell** where there are, among other things, natural ways to endow a cell-set with monoid and comonoid object structure. This is achieved using the category of pointed cell-sets, which we denote by **PCell**.

Definition 2.2.9. The objects of **PCell** are pointed cell-sets, and a morphism between pointed cell-sets is a cell-map $f \colon \mathscr{C} \longrightarrow \mathscr{D}$, termed *pointed cell-map*, that maps base points to base points in a very particular way: if $x \in 1_{\mathscr{C}}$ is a base point, then the multiset $f(x)$ consists of exactly one base point of $\mathscr{D}$ with multiplicity one. Unless otherwise stated, a cell-map between pointed cell-sets will always be considered pointed.

Remark 2.2.10. In [RS98], a pointed cell-map is defined as a cell-map that maps base points to base points without further conditions on the cardinality of the image of a base point. The reason why we modify their definition is that, in our context of multisets, such a modification is needed to ensure several properties whose veracity is claimed by them, but that fail to be true without this additional assumption. We will indicate these properties throughout the work. Moreover, even if we consider this modified definition, there are some assertions that do not hold as they claim. For instance, in [RS98] the trivial cell-set $\mathscr{T}$ is claimed to be terminal within the category **PCell**. However, the following reasoning shows the opposite: let $\mathscr{C}$ be any pointed cell-set. A pointed cell-map $f \colon \mathscr{C} \longrightarrow \mathscr{T}$ may be defined as follows: if $x \in 1_{\mathscr{C}}$ is any base point of $\mathscr{C}$, then x must be mapped to the submultiset $\{*\}$, where $*$ appears with multiplicity one. If $x \in \mathscr{C}_n$, with $n \neq 0$ is any element of non-zero dimension, by the dimension-preserving property it must be mapped to the empty set. But if $\mathscr{C}$ is not connected, there must exist a zero-dimensional element

$x \in \mathscr{C}_0 \setminus 1_{\mathscr{C}}$ which is not a base point, and all elements from its defining equivalence class $[x]$ have the freedom of being mapped to the empty set, or to the point $\{*\}$ with any multiplicity. This fact shows that $\mathscr{T}$ cannot be terminal unless $\mathscr{C}$ is connected.

Sometimes asking equality between two maps, or objects in general, is a too rigid condition. For our purpose, it is enough to consider cell-maps up to equivalence, in some sense that we will define.

Definition 2.2.11. We say that two cell-maps $f, g \colon \mathscr{C} \longrightarrow \mathscr{D}$ are *equivalent*, and we denote it by $f \sim g$, if $f(x) \sim g(x)$ for every $x \in \mathscr{C}$.

Our definition of multisets also ensures that composition of cell-maps modulo equivalence is well-defined.

Proposition 2.2.12. *Let*

$$\mathscr{C} \underset{g'}{\overset{g}{\rightrightarrows}} \mathscr{D} \underset{f'}{\overset{f}{\rightrightarrows}} \mathscr{E}$$

be a diagram in **Cell**. *Suppose that $g \sim g'$ and $f \sim f'$. Then, we also have that*

$$f \circ g \sim f' \circ g'.$$

Proof. Let $x \in \mathscr{C}$. Since $g \sim g'$, we have that

$$(y_i)_{i \in I} = g(x) \sim g(x') = (y'_j)_{j \in J},$$

which means that there exists a bijection $\varphi \colon I \longrightarrow J$ such that, for every $i \in I$,

$$y_i \sim y'_{\varphi(i)}.$$

For each $i \in I$, we have that

$$f(y_i) \sim f(y'_{\varphi(i)}) \text{ and } f'(y_i) \sim f'(y'_{\varphi(i)}),$$

just because f and f' are cell-maps. Moreover, using that $f \sim f'$, we have the following equivalence square

$$\begin{array}{ccc} f(y_i) & \sim & f(y'_{\varphi(i)}) \\ \wr & & \wr \\ f'(y_i) & \sim & f'(y'_{\varphi(i)}), \end{array}$$

so by transitivity we deduce that

$$f(y_i) \sim f'(y'_{\varphi(i)})$$

for every $i \in I$. The rest of the proof is analogous to the one done in 2.2.8. ∎

As a corollary of the above proposition, we can define the categories **Cell**$'$ and **PCell**$'$ by retaining the original objects but defining morphisms as equivalence classes of cell-maps and pointed cell-maps, respectively. Isomorphic cell-sets in these categories are termed equivalent.

Definition 2.2.13. We call two cell-sets $\mathscr{C}$ and $\mathscr{D}$ *equivalent* if there exist cell-maps

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "f"] & \mathscr{D} \arrow[l, shift left, "g"] \end{tikzcd}$$

such that $g \circ f \sim \mathrm{id}_{\mathscr{C}}$ and $f \circ g \sim \mathrm{id}_{\mathscr{D}}$.

Now that we know enough information about the category **Cell**, the next step is to define a functor

$$Z_* \colon \mathbf{Cell} \longrightarrow \mathbf{GAb},$$

where **GAb** is the category of $\mathbb{N}$-graded abelian groups, that transforms the combinatorial structure encoded in cell-sets to algebraic structure in the form of graded abelian groups. That is, we equip **Cell** with a symmetric monoidal structre such that Z_* becomes a braided strong colax monoidal functor, considering the usual tensor product of $\mathbb{Z}$-modules in **GAb**. This construction enables us to produce graded $\mathbb{Z}$-modules with the classical structures of algebra, coalgebra, bialgebra an Hopf algebra, starting from monoid, comonoid, bimonoid and Hopf monoid objects in the category of cell-sets, respectively. Moreover, the functor Z_* is defined in such a way that it factors through **Cell′**

$$\begin{tikzcd} \mathbf{Cell} \arrow[rr, "Z_*"] \arrow[dr] & & \mathbf{GAb} \\ & \mathbf{Cell}' \arrow[ur, dashed] & \end{tikzcd}$$

in a unique way. This is important, since it implies that we can relax the compatibility conditions imposed on these objects derived from the monoidal structure of **Cell** without changing the obtained algebraic object.

Definition 2.2.14. For every cell-set $\mathscr{C}$ and every $n \in \mathbb{Z}$, we denote by $Z_n(\mathscr{C})$ the free abelian group on the set $\mathscr{C}_n/\sim$ of equivalence classes of n-dimensional elements of $\mathscr{C}$, and we denote by $Z_*(\mathscr{C})$ the graded abelian group defined by the sequence of abelian groups $\{Z_n(\mathscr{C})\}_{n\in\mathbb{Z}}$. Equivalently, $Z_*(\mathscr{C})$ is the free abelian group on the quotient set $\mathscr{C}/\sim$. We call $Z_*(\mathscr{C})$, making an abuse of language, the *free abelian group on* $\mathscr{C}$. This construction can be done over any commutative ring R with unit, so that $R_*(\mathscr{C})$ denotes the free R-module on $\mathscr{C}$ defined in the same way as before.

Notice that every cell-map $f \colon \mathscr{C} \longrightarrow \mathscr{D}$ defines a morphism f_* of graded abelian groups, which is determined by

$$\begin{array}{rcl} f_* \colon Z_*(\mathscr{C}) & \longrightarrow & Z_*(\mathscr{D}) \\ {[x]} & \longmapsto & \sum_{y\in f(x)} [y]\,. \end{array}$$

Again, the sum runs over all elements of $f(x)$ counting multiplicities: if $f(x) = (y_i)_{i\in I}$, then

$$\sum_{y\in f(x)} [y] := \sum_{i\in I} [y_i].$$

Proposition 2.2.15. *The above-mentioned assignments define a functor*

$$Z_* \colon \mathbf{Cell} \longrightarrow \mathbf{GAb}.$$

Proof. First of all, before asking ourselves if such a construction is functorial, we must check that every morphism f_* induced by a cell-map f is well-defined as a map. Let f be a cell-map and consider its associated morphism f_*. Let $x, x' \in \mathscr{C}$ such that $x \sim x'$. Then, since f is a cell-map, we have that

$$(y_i)_{i\in I} = f(x) \sim f(x') = (y_i')_{j\in J},$$

which means that there exists a bijection

$$\varphi \colon I \longrightarrow J$$

such that

$$y_i \sim y'_{\varphi(i)}$$

for every $i \in I$. This clearly implies that $f_*([x]) = f_*([x'])$. Indeed,

$$f_*([x]) = \sum_{i\in I}[y_i] = \sum_{i\in I}[y'_{\varphi(i)}] = \sum_{j\in J}[y'_j] = f_*([x']).$$

Thus, f_* is well-defined. Now let us check functoriality: for every $\mathscr{C} \in \mathbf{Cell}$, it is trivially true that

$$(\mathrm{id}_{\mathscr{C}})_* = \mathrm{id}_{Z_*(\mathscr{C})}\,.$$

If we have a sequence of composable cell-maps

$$\mathscr{C} \xrightarrow{g} \mathscr{D} \xrightarrow{f} \mathscr{E},$$

we want to see that

$$(f \circ g)_* = f_* \circ g_*.$$

For each $x \in \mathscr{C}$, write $g(x) = (y_i)_{i\in I}$ and, for every $i \in I$, put $f(y_i) = (a_k^i)_{k\in K_i}$. Then,

$$(f \circ g)_*([x]) = \sum_{i\in I}\sum_{k\in K_i}[a_k^i] = \sum_{i\in I} f_*([y_i]) = f_*\left(\sum_{i\in I}[y_i]\right) = (f_* \circ g_*)([x]).$$

∎

Now that we have our well-defined functor Z_*, we realize why defining maps up to equivalence is a good idea.

Proposition 2.2.16. *Let $f, g \colon \mathscr{C} \longrightarrow \mathscr{D}$ be two cell-maps. Then,*

$$f \sim g \iff f_* = g_*.$$

In other words, Z_* factors through the canonical functor $\mathbf{Cell} \longrightarrow \mathbf{Cell}'$

$$\begin{tikzcd} \mathbf{Cell} \arrow[rr, "Z_*"] \arrow[dr] & & \mathbf{GAb} \\ & \mathbf{Cell}', \arrow[ur, dashed, "\exists!\, \widetilde{Z_*}"'] & \end{tikzcd}$$

with $\widetilde{Z_*}$ being faithful.

Proof. Assume that $f \sim g$. Then, for every $x \in \mathscr{C}$, we have that

$$(y_i)_{i\in I} = f(x) \sim g(x) = (y'_j)_{j\in J},$$

which means that there exists a bijection $\varphi\colon I \longrightarrow J$ such that, for each $i \in I$,

$$y_i \sim y'_{\varphi(i)}.$$

Then,

$$f_*([x]) = \sum_{i\in I}[y_i] = \sum_{i\in I}[y'_{\varphi(i)}] = \sum_{j\in J}[y'_j] = g_*([x]).$$

Conversely, suppose that $f_*([x]) = g_*([x])$ for every $x \in \mathscr{C}$. Then,

$$f_*([x]) = \sum_{i\in I}[y_i] = \sum_{j\in J}[y'_j] = g_*([x]),$$

which means that the sequences they define in the free abelian group $Z_*(\mathscr{C})$ are equal. By definition, this implies that $f(x) \sim g(x)$, as desired. ∎

In order to define algebras, coalgebras, bialgebras and Hopf algebras through Z_*, we first need to be able to endow **Cell** with some appropriate symmetric monoidal structure such that it is respected by Z_*. Although it will not be needed in our work, we start by checking how does the coproduct in **Cell** look like. It is also interesting since it is respected by the free abelian group functor in a certain way.

Proposition 2.2.17. *Given two cell-sets $\mathscr{C}$ and $\mathscr{D}$, we can take as a model for their coproduct $\mathscr{C} \amalg \mathscr{D}$ in* **Cell**, *also called their disjoint union, the disjoint union*

$$\mathscr{C} \amalg \mathscr{D} = \mathscr{C} \times \{1\} \cup \mathscr{D} \times \{2\}$$

of their underlying sets, together with the natural equivalence relation, that is,

$$x \sim_{\mathscr{C}\amalg\mathscr{D}} y \iff \begin{cases} x, y \in \mathscr{C} \text{ and } x \sim_{\mathscr{C}} y, \\ x, y \in \mathscr{D} \text{ and } x \sim_{\mathscr{D}} y, \end{cases}$$

and with dimension function given by the map $d_{\mathcal{C}\amalg\mathcal{D}}\colon C \amalg D \longrightarrow \mathbb{Z}$ defined as

$$\begin{aligned} d_{\mathcal{C}\amalg\mathcal{D}}\colon C \amalg D &\longrightarrow \mathbb{Z} \\ x &\longmapsto d_{\mathcal{C}\amalg\mathcal{D}}(x) := \begin{cases} d_{\mathscr{C}}(x), & \text{if } x \in \mathscr{C}, \\ d_{\mathscr{D}}(x), & \text{if } x \in \mathscr{D}. \end{cases} \end{aligned}$$

Moreover, there is an isomorphism

$$Z_*(\mathscr{C} \amalg \mathscr{D}) \cong Z_*(\mathscr{C}) \oplus Z_*(\mathscr{D}).$$

Proof. Consider the following diagram in **Cell**

$$\begin{tikzcd} & \mathscr{D} \arrow[d, "i_{\mathscr{D}}"] \\ \mathscr{C} \arrow[r, "i_{\mathscr{C}}"'] & \mathscr{C} \amalg \mathscr{D}, \end{tikzcd}$$

where $i_{\mathscr{C}}$ and $i_{\mathscr{D}}$ are the obvious cell-maps

$$\begin{aligned} i_{\mathscr{C}} \colon \mathscr{C} &\longrightarrow \mathscr{C} \amalg \mathscr{D} \\ x &\longmapsto \{(x,1)\} \end{aligned}$$

and

$$\begin{aligned} i_{\mathscr{D}} \colon \mathscr{D} &\longrightarrow \mathscr{C} \amalg \mathscr{D} \\ x &\longmapsto \{(x,2)\}\,. \end{aligned}$$

Given any cell-set $\mathscr{E}$ and any pair of cell-maps

$$\begin{tikzcd} & \mathscr{D} \arrow[d, "i_{\mathscr{D}}"] \arrow[ddr, bend left, "g"] & \\ \mathscr{C} \arrow[r, "i_{\mathscr{C}}"'] \arrow[drr, bend right, "f"'] & \mathscr{C} \amalg \mathscr{D} & \\ & & \mathscr{E} \end{tikzcd}$$

as in the diagram, we define

$$\begin{aligned} \varphi \colon C \amalg D &\longrightarrow \mathscr{E} \\ x &\longmapsto \varphi(x) := \begin{cases} f(x), & \textit{if } x \in \mathscr{C}, \\ g(x), & \textit{if } x \in \mathscr{D}. \end{cases} \end{aligned}$$

Clearly φ is a cell-map, and it is the unique one that makes the following diagram commutative in **Cell**

$$\begin{tikzcd} & \mathscr{D} \arrow[d, "i_{\mathscr{D}}"] \arrow[ddr, bend left, "g"] & \\ \mathscr{C} \arrow[r, "i_{\mathscr{C}}"'] \arrow[drr, bend right, "f"'] & \mathscr{C} \amalg \mathscr{D} \arrow[dr, dashed, "\varphi"] & \\ & & \mathscr{E}\,. \end{tikzcd}$$

Finally, the isomorphism

$$Z_*(\mathscr{C} \amalg \mathscr{D}) \cong Z_*(\mathscr{C}) \oplus Z_*(\mathscr{D})$$

is obvious from the fact that

$$(\mathscr{C} \amalg \mathscr{D})_n = \mathscr{C}_n \amalg \mathscr{D}_n.$$

■

However, for the product case, we are interested in a binary operation that does not coincide with the categorical product in **Cell**. The operation is defined as follows: given two cell-sets $\mathscr{C}$ and $\mathscr{D}$, their *direct product* is the cell-set defined by the cartesian product $\mathscr{C} \times \mathscr{D}$ of their underlying sets, together with the natural equivalence relation defined componentwise, that is,

$$(x,y) \sim_{\mathscr{C}\times\mathscr{D}} (x',y') \iff x \sim_{\mathscr{C}} x' \text{ and } y \sim_{\mathscr{D}} y',$$

and with dimension function given by $d_{\mathscr{C}\times\mathscr{D}}(x,y) := d_{\mathscr{C}}(x)+d_{\mathscr{D}}(y)$ for every $(x,y) \in \mathscr{C} \times \mathscr{D}$. From this, it is clear that

$$(\mathscr{C} \times \mathscr{D})_n = \bigcup_{k\in\mathbb{Z}} (\mathscr{C}_k \times \mathscr{D}_{n-k}),$$

so there is an isomorphism

$$\begin{array}{rcl} Z_*(\mathscr{C} \times \mathscr{D}) & \longrightarrow & Z_*(\mathscr{C}) \otimes Z_*(\mathscr{D}) \\ [(x,y)] & \longmapsto & [x] \otimes [y] \,. \end{array}$$

Now if $f\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $f'\colon \mathscr{C}' \longrightarrow \mathscr{D}'$ are cell-maps, their *direct product* $f \times f'$ is defined in the natural way

$$\begin{array}{rcl} f \times f'\colon \mathscr{C} \times \mathscr{C}' & \longrightarrow & \mathscr{D} \times \mathscr{D}' \\ (x,y) & \longmapsto & (f \times f')(x,y) := f(x) \times f'(y), \end{array}$$

where the product $f(x) \times f'(y)$ is the product of submultisets, which by definition gives a submultiset of $\mathscr{D} \times \mathscr{D}'$. This construction gives the desired symmetric monoidal structure in **Cell** that we were looking for.

Proposition 2.2.18. *The assignment*

$$\begin{array}{rcl} \times\colon \mathbf{Cell} \times \mathbf{Cell} & \longrightarrow & \mathbf{Cell} \\ (\mathscr{C},\mathscr{D}) & \longmapsto & \mathscr{C} \times \mathscr{D} \\ (f,f') & \longmapsto & f \times f' \end{array}$$

is a functor and it equips **Cell** *with a symmetric monoidal structure, with unit given by the trivial cell-set* $\mathscr{T}$*, braiding given by the switch cell-map*

$$\begin{array}{rcl} \tau = \tau_{\mathscr{C}\times\mathscr{D}}\colon \mathscr{C} \times \mathscr{D} & \longrightarrow & \mathscr{D} \times \mathscr{C} \\ (x,y) & \longmapsto & \{(y,x)\}\,, \end{array}$$

and associator, left and right unitors given by the natural isomorphisms of cell-sets

$$\begin{array}{rcl} a = a_{\mathscr{C},\mathscr{D},\mathscr{E}}\colon (\mathscr{C} \times \mathscr{D}) \times \mathscr{E} & \longrightarrow & \mathscr{C} \times (\mathscr{D} \times \mathscr{E}) \\ ((x,y),z) & \longmapsto & \{(x,(y,z))\}\,, \end{array}$$

$$\begin{array}{rcl} \lambda = \lambda_{\mathscr{C}}\colon \mathscr{T} \times \mathscr{C} & \longrightarrow & \mathscr{C} \\ (*,x) & \longmapsto & \{x\}\,, \end{array}$$

and

$$\begin{array}{rcl} \rho = \rho_{\mathscr{C}}\colon \mathscr{C} \times \mathscr{T} & \longrightarrow & \mathscr{C} \\ (x,*) & \longmapsto & \{x\}\,, \end{array}$$

respectively.

Now that we have defined our symmetric monoidal structure on **Cell**, we show that the free abelian group functor

$$Z_*\colon (\mathbf{Cell}, \times, \mathscr{T}, \tau) \longrightarrow (\mathbf{GAb}, \otimes, \mathbb{Z}, B)$$

is a braided costrong monoidal functor, where the tensor product structure in **GAb** is the natural one.

Proposition 2.2.19. *The free abelian group functor*

$$Z_*\colon (\mathbf{Cell}, \times, \mathscr{T}, \tau) \longrightarrow (\mathbf{GAb}, \otimes, \mathbb{Z}, B)$$

is a braided costrong monoidal functor, with structure maps given by

$$\begin{array}{rcl} \psi = \psi_{\mathscr{C},\mathscr{D}}\colon Z_*(\mathscr{C} \times \mathscr{D}) & \longrightarrow & Z_*(\mathscr{C}) \otimes Z_*(\mathscr{D}) \\ [(c,d)] & \longmapsto & [c] \otimes [d] \end{array} \tag{2.2}$$

and

$$\begin{array}{rcl} \psi_0\colon Z_*(\mathscr{T}) & \longrightarrow & \mathbb{Z} \\ [*] & \longmapsto & 1. \end{array} \tag{2.3}$$

Proof. Firstly, we start by checking that 2.2 is a natural isomorphism. We already know that each component is an isomorphism, so we just need to check that it defines a natural transformation between the functors

$$\begin{tikzcd} \mathbf{Cell} \times \mathbf{Cell} \arrow[rr, bend left, "Z_* \circ \times"] \arrow[rr, bend right, "\otimes \circ (Z_* \times Z_*)"'] \arrow[rr, phantom, "\Downarrow \psi"] & & \mathbf{GAb}. \end{tikzcd}$$

Indeed, let

$$f\colon \mathscr{C} \longrightarrow \mathscr{D} \quad \text{and} \quad g\colon \mathscr{E} \longrightarrow \mathscr{F}$$

be a pair of cell-maps. We have to check that the following diagram

$$\begin{tikzcd} Z_*(\mathscr{C} \times \mathscr{E}) \arrow[r, "\psi_{\mathscr{C},\mathscr{E}}"] \arrow[d, "(f\times g)_*"'] & Z_*(\mathscr{C}) \otimes Z_*(\mathscr{E}) \arrow[d, "f_* \otimes g_*"] \\ Z_*(\mathscr{D} \times \mathscr{F}) \arrow[r, "\psi_{\mathscr{D},\mathscr{F}}"'] & Z_*(\mathscr{D}) \otimes Z_*(\mathscr{F}) \end{tikzcd} \tag{2.4}$$

is commutative. For every $(c, e) \in \mathscr{C} \times \mathscr{E}$, write

$$f(c) = (a_i)_{i \in I} \quad \text{and} \quad g(e) = (b_j)_{j \in J}.$$

On the one hand, we have that

$$(f_* \otimes g_*)(\psi_{\mathscr{C},\mathscr{E}}([(c,e)])) = (f_* \otimes g_*)([c] \otimes [e]) = f_*([c]) \otimes g_*([e]) =$$

$$\left(\sum_{i \in I} [a_i]\right) \otimes \left(\sum_{j \in J} [b_j]\right) = \sum_{(i,j) \in I \times J} [a_i] \otimes [b_j].$$

On the other hand,

$$\psi_{\mathscr{D},\mathscr{F}}((f \times g)_*([(c,e)])) = \sum_{(i,j) \in I \times J} \psi_{\mathscr{D},\mathscr{F}}([(a_i, b_j)]) = \sum_{(i,j) \in I \times J} [a_i] \otimes [b_j].$$

Therefore
$$(f_* \otimes g_*) \circ \psi_{\mathscr{C},\mathscr{E}} = \psi_{\mathscr{D},\mathscr{F}} \circ (f \times g)_*,$$
so the diagram 2.4 is commutative. The map 2.3 is obviously an isomorphism of graded abelian groups, so it remains to check the commutativity of four diagrams. We start with the following diagram

$$\begin{tikzcd}
(Z_*(\mathscr{C}) \otimes Z_*(\mathscr{D})) \otimes Z_*(\mathscr{E}) \arrow[r, "a"] & Z_*(\mathscr{C}) \otimes (Z_*(\mathscr{D}) \otimes Z_*(\mathscr{E})) \\
Z_*(\mathscr{C} \times \mathscr{D}) \otimes Z_*(\mathscr{E}) \arrow[u, "\psi_{\mathscr{C},\mathscr{D}} \otimes \mathrm{id}_{Z_*(\mathscr{E})}"] & Z_*(\mathscr{C}) \otimes Z_*(\mathscr{D} \times \mathscr{E}) \arrow[u, "\mathrm{id}_{Z_*(\mathscr{C})} \otimes \psi_{\mathscr{D},\mathscr{E}}"'] \\
Z_*((\mathscr{C} \times \mathscr{D}) \times \mathscr{E}) \arrow[r, "Z_*(a)"'] \arrow[u, "\psi_{\mathscr{C}\times\mathscr{D},\mathscr{E}}"] & Z_*(\mathscr{C} \times (\mathscr{D} \times \mathscr{E})), \arrow[u, "\psi_{\mathscr{C},\mathscr{D}\times\mathscr{E}}"']
\end{tikzcd}$$

whose commutativity has to be checked for every triple of cell-sets $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$. For every $((c,d),e) \in \mathscr{C} \times (\mathscr{D} \times \mathscr{E})$,

$$\mathrm{id}_{Z_*(\mathscr{C})} \otimes \psi_{\mathscr{D},\mathscr{E}}(\psi_{\mathscr{C},\mathscr{D}\times\mathscr{E}}(Z_*(a)([((c,d),e)]))) = \mathrm{id}_{Z_*(\mathscr{C})} \otimes \psi_{\mathscr{D},\mathscr{E}}(\psi_{\mathscr{C},\mathscr{D}\times\mathscr{E}}([(c,(d,e))])) =$$
$$\mathrm{id}_{Z_*(\mathscr{C})} \otimes \psi_{\mathscr{D},\mathscr{E}}([c] \otimes [(d,e)]) = [c] \otimes ([d] \otimes [e]) = a(([c] \otimes [d]) \otimes [e]) =$$
$$a(\psi_{\mathscr{C},\mathscr{D}} \otimes \mathrm{id}_{Z_*(\mathscr{E})}([(c,d)] \otimes [e])) = a(\psi_{\mathscr{C},\mathscr{D}} \otimes \mathrm{id}_{Z_*(\mathscr{E})}(\psi_{\mathscr{C}\times\mathscr{D},\mathscr{E}}([((c,d),e)]))).$$

Therefore,
$$(\mathrm{id}_{Z_*(\mathscr{C})} \otimes \psi_{\mathscr{D},\mathscr{E}}) \circ \psi_{\mathscr{C},\mathscr{D}\times\mathscr{E}} \circ Z_*(a) = a \circ (\psi_{\mathscr{C},\mathscr{D}} \otimes \mathrm{id}_{Z_*(\mathscr{E})}) \circ \psi_{\mathscr{C}\times\mathscr{D},\mathscr{E}},$$

which means that the above diagram is commutative. Furthermore, we have to show that this pair of diagrams

$$\begin{tikzcd}
\mathbb{Z} \otimes Z_*(\mathscr{C}) & Z_*(\mathscr{C}) \arrow[l, "\lambda^{-1}_{Z_*(\mathscr{C})}"'] \arrow[d, "Z_*(\lambda^{-1}_{\mathscr{C}})"] \\
Z_*(\mathscr{T}) \otimes Z_*(\mathscr{C}) \arrow[u, "\psi_0 \otimes \mathrm{id}_{Z_*(\mathscr{C})}"] & Z_*(\mathscr{T} \times \mathscr{C}) \arrow[l, "\psi_{\mathscr{T},\mathscr{C}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_*(\mathscr{C}) \otimes \mathbb{Z} & Z_*(\mathscr{C}) \arrow[l, "\rho^{-1}_{Z_*(\mathscr{C})}"'] \arrow[d, "Z_*(\rho^{-1}_{\mathscr{C}})"] \\
Z_*(\mathscr{C}) \otimes Z_*(\mathscr{T}) \arrow[u, "\mathrm{id}_{Z_*(\mathscr{C})} \otimes \psi_0"] & Z_*(\mathscr{C} \times \mathscr{T}) \arrow[l, "\psi_{\mathscr{C},\mathscr{T}}"]
\end{tikzcd}$$

are commutative for every cell-set $\mathscr{C}$. Let us show, for instance, that the left square commutes, and the right one is left as an exercise since the proof is completely analogous. For every $c \in \mathscr{C}$,

$$\psi_0 \otimes \mathrm{id}_{Z_*(\mathscr{C})}(\psi_{\mathscr{T},\mathscr{C}}(Z_*(\lambda^{-1}_{\mathscr{C}})([c]))) = \psi_0 \otimes \mathrm{id}_{Z_*(\mathscr{C})}(\psi_{\mathscr{T},\mathscr{C}}([(*,c)])) = \psi_0 \otimes \mathrm{id}_{Z_*(\mathscr{C})}([*] \otimes [c]) =$$
$$1 \otimes [c] = \lambda^{-1}_{Z_*(\mathscr{C})}([c]),$$

which means that

$$(\psi_0 \otimes \mathrm{id}_{Z_*(\mathscr{C})}) \circ \psi_{\mathscr{T},\mathscr{C}} \circ Z_*(\lambda_{\mathscr{C}}^{-1}) = \lambda_{Z_*(\mathscr{C})}^{-1}.$$

Thus, the left square commutes. This argument shows that Z_* together with the structure maps defines a costrong monoidal functor. To see that it is braided, we just have to check that this last diagram commutes

$$\begin{tikzcd}
Z_*(\mathscr{C} \times \mathscr{D}) \arrow[r, "\psi_{\mathscr{C},\mathscr{D}}"] \arrow[d, "Z_*(\tau)"'] & Z_*(\mathscr{C}) \otimes Z_*(\mathscr{D}) \arrow[d, "B"] \\
Z_*(\mathscr{D} \times \mathscr{C}) \arrow[r, "\psi_{\mathscr{D},\mathscr{C}}"'] & Z_*(\mathscr{D}) \otimes Z_*(\mathscr{C}).
\end{tikzcd}$$

It is completely obvious that it does so, but let us check it: given any $(c, d) \in \mathscr{C} \times \mathscr{D}$,

$$B(\psi_{\mathscr{C},\mathscr{D}}([(c, d)])) = B([c] \otimes [d]) = [d] \otimes [c] = \psi_{\mathscr{D},\mathscr{C}}([(d, c)]) = \psi_{\mathscr{D},\mathscr{C}}(Z_*(\tau)([(c, d)])).$$

Hence the above diagram commutes, and therefore Z_* is a braided strong colax monoidal functor, as desired. ■

2.3 Product and Coproduct Structures in Cell

Now that we know that the free abelian group functor is strong monoidal, we can translate monoid, comonoid, bimonoid and Hopf monoid objects in **Cell** to classical algebras, coalgebras, bialgebras and Hopf algebras in **GAb**. Moreover, since Z_* factors through the canonical functor $\mathbf{Cell} \longrightarrow \mathbf{Cell}'$, we only need to consider such objects in their relaxed version within $\mathbf{Cell}'$.

In this section, we give sufficient conditions in order to have such monoidal structures in the relaxed category $\mathbf{Cell}'$. We also remark the mathematical inconsistencies made in [RS98], and we propose alternative solutions for each case.

Remark 2.3.1. Whenever we have pointed cell-sets $\mathscr{C}$ and $\mathscr{D}$ with base points $1_{\mathscr{C}}$ and $1_{\mathscr{D}}$, respectively, we always assume that the direct product $\mathscr{C} \times \mathscr{D}$ is pointed, with $1_{\mathscr{C}\times\mathscr{D}} := 1_{\mathscr{C}} \times 1_{\mathscr{D}}$, unless otherwise stated.

Definition 2.3.2. Let $\mathscr{C}$ be a cell-set. A *product* on $\mathscr{C}$ is a cell-map

$$\mu \colon \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

which is associative up to equivalence, that is, the diagram

$$\begin{tikzcd}
\mathscr{C} \times \mathscr{C} \arrow[r, "\mu"] & \mathscr{C} \\
\mathscr{C} \times \mathscr{C} \times \mathscr{C} \arrow[u, "\mathrm{id}_{\mathscr{C}} \times \mu"] \arrow[r, "\mu \times \mathrm{id}_{\mathscr{C}}"'] & \mathscr{C} \times \mathscr{C} \arrow[u, "\mu"']
\end{tikzcd}$$

commutes up to equivalence, which means that

$$\mu \circ (\mathrm{id}_{\mathscr{C}} \times \mu) \sim \mu \circ (\mu \times \mathrm{id}_{\mathscr{C}}).$$

Proposition 2.3.3. *Let $\mathscr{C}$ be a pointed cell-set equipped with a product*

$$\mu\colon \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}.$$

Assume that this product verifies that

$$\mu(u, x) \sim \{x\} \sim \mu(x, u) \tag{2.5}$$

for all $x \in \mathscr{C}$ and all $u \in 1_{\mathscr{C}}$. Then, given any pointed cell-map $\eta\colon \mathscr{T} \longrightarrow \mathscr{C}$, the triple $(\mathscr{C}, \mu, \eta)$ defines a monoid object in **Cell**$'$.

Proof. We just need to check that, given any pointed cell-map

$$\eta\colon \mathscr{T} \longrightarrow \mathscr{C},$$

the diagram

$$\begin{tikzcd}
 & \mathscr{C} \times \mathscr{C} \arrow[dd, "\mu"] & \\
\mathscr{T} \times \mathscr{C} \arrow[ur, "\eta \times \mathrm{id}_{\mathscr{C}}"] \arrow[dr, "\lambda_{\mathscr{C}}"'] & & \mathscr{C} \times \mathscr{T} \arrow[ul, "\mathrm{id}_{\mathscr{C}} \times \eta"'] \arrow[dl, "\rho_{\mathscr{C}}"] \\
 & \mathscr{C} &
\end{tikzcd}$$

commutes up to equivalence of cell-maps, but we observe that the commutativity of this diagram follows immediately using the hypothesis 2.5. ■

Remark 2.3.4. This proposition is an example of why do we need to modify the definition of pointed cell-map (see 2.2.9). If we were using the definition given in [RS98], the condition on the product would not be enough to ensure that every pointed cell-map $\eta\colon \mathscr{T} \longrightarrow \mathscr{C}$ is a unit. We also want to emphasize that the product μ does not need to be a pointed cell-map.

Definition 2.3.5. A cell-map $f\colon (\mathscr{C}, \mu_{\mathscr{C}}) \longrightarrow (\mathscr{D}, \mu_{\mathscr{D}})$ is termed *multiplicative* if it commutes with products up to equivalence, that is, the diagram

$$\begin{tikzcd}
\mathscr{C} \times \mathscr{C} \arrow[r, "f \times f"] \arrow[d, "\mu_{\mathscr{C}}"'] & \mathscr{D} \times \mathscr{D} \arrow[d, "\mu_{\mathscr{D}}"] \\
\mathscr{C} \arrow[r, "f"'] & \mathscr{D}
\end{tikzcd}$$

commutes up to equivalence of cell-maps.

A pointed multiplicative cell-map between monoid objects as in proposition 2.3.3 is automatically a morphism of monoid objects in **Cell**$'$.

Proposition 2.3.6. *Let $(\mathscr{C}, \mu_{\mathscr{C}}, \eta_{\mathscr{C}})$ and $(\mathscr{D}, \mu_{\mathscr{D}}, \eta_{\mathscr{D}})$ be monoid objects as in 2.3.3. Then, any pointed cell-map*

$$f\colon (\mathscr{C}, \mu_{\mathscr{C}}, \eta_{\mathscr{C}}) \longrightarrow (\mathscr{D}, \mu_{\mathscr{D}}, \eta_{\mathscr{D}})$$

is a morphism of monoid objects in **Cell**$'$.

Proof. We just need to check that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "f"] & & \mathscr{D} \\ & \mathscr{T} \arrow[ul, "\eta_{\mathscr{C}}"] \arrow[ur, "\eta_{\mathscr{D}}"'] & \end{tikzcd}$$

is commutative up to equivalence. We have that both $f(\eta_{\mathscr{T}}(*))$ and $\eta_{\mathscr{D}}(*)$ are base points of $\mathscr{D}$ by hypothesis, therefore

$$f(\eta_{\mathscr{T}}(*)) \sim \eta_{\mathscr{D}}(*),$$

as claimed. ∎

Now we are going to do something similar with the dual concept of the product, which is the coproduct.

Definition 2.3.7. Let $\mathscr{C}$ be a cell-set. A *coproduct* on $\mathscr{C}$ is a cell-map

$$\delta\colon \mathscr{C} \longrightarrow \mathscr{C} \times \mathscr{C}$$

which is coassociative up to equivalence, that is, the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathscr{C} \times \mathscr{C} \arrow[d, "\delta \times \mathrm{id}_{\mathscr{C}}"] \\ \mathscr{C} \times \mathscr{C} \arrow[r, "\mathrm{id}_{\mathscr{C}} \times \delta"'] & \mathscr{C} \times \mathscr{C} \times \mathscr{C} \end{tikzcd}$$

commutes up to equivalence, which means that

$$(\mathrm{id}_{\mathscr{C}} \times \delta) \circ \delta \sim (\delta \times \mathrm{id}_{\mathscr{C}}) \circ \delta.$$

Proposition 2.3.8. *Let $\mathscr{C}$ be a pointed cell-set with a coproduct*

$$\delta\colon \mathscr{C} \longrightarrow \mathscr{C} \times \mathscr{C}.$$

Let $\varepsilon\colon \mathscr{C} \longrightarrow \mathscr{T}$ be the pointed cell-map given by

$$\begin{aligned} \varepsilon\colon \mathscr{C} &\longrightarrow \mathscr{T} \\ x &\longmapsto \varepsilon(x) := \begin{cases} \{*\}, & \textit{if } x \in 1_{\mathscr{C}}, \\ \varnothing, & \textit{otherwise.} \end{cases} \end{aligned}$$

Assume that for every $x \in \mathscr{C}$, the multiset $\delta(x)$ contains elements of the form (u, x_1) and (x_2, v), with $u, v \in 1_{\mathscr{C}}$ and $x \sim x_1 \sim x_2$. Moreover, these elements must be unique in the following sense:

1. *The elements (u, x_1) and (x_2, v) must have multiplicity one.*

2. *If there exists some $(y_1, y_2) \in \delta(x)$ with $y_1 \in 1_{\mathscr{C}}$, then $y_1 = u$, $y_2 = x_1$.*

3. *Analogously, if there exists some $(y_1, y_2) \in \delta(x)$ with $y_2 \in 1_{\mathscr{C}}$, then $y_1 = x_2$, $y_2 = v$.*

If that is the case, then the triple $(\mathscr{C}, \delta, \varepsilon)$ defines a comonoid object in **Cell′**.

Proof. We just need to show that the diagram

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, "\lambda_{\mathscr{C}}^{-1}"'] \arrow[dr, "\rho_{\mathscr{C}}^{-1}"] \arrow[dd, "\delta"] & \\ \mathscr{T} \times \mathscr{C} & & \mathscr{C} \times \mathscr{T} \\ & \mathscr{C} \times \mathscr{C} \arrow[ul, "\varepsilon \times \mathrm{id}_{\mathscr{C}}"] \arrow[ur, "\mathrm{id}_{\mathscr{C}} \times \varepsilon"'] & \end{tikzcd}$$

commutes up to equivalence of cell-maps. First, let us focus on the left-hand side triangle. This triangle commutes (up to equivalence) iff for every $x \in \mathscr{C}$, the multisets

$$(\varepsilon \times \mathrm{id}_{\mathscr{C}})(\delta(x)) \sim \{(*, x)\}$$

are equivalent. By definition, this just means that there exists some $x' \sim x$ such that

$$(\varepsilon \times \mathrm{id}_{\mathscr{C}})(\delta(x)) = \{(*, x')\}\,.$$

By hypothesis, $\delta(x)$ contains an element of the form (u, x_1), with $u \in 1_{\mathscr{C}}$ and $x_1 \sim x$, so we can choose $x' := x_1$. Now, let us look at

$$(\varepsilon \times \mathrm{id}_{\mathscr{C}})(\delta(x)) = \bigcup_{(y_1,y_2)\in\delta(x)} \varepsilon(y_1) \times \{y_2\}\,.$$

We distinguish 2 cases: if $(y_1, y_2) \in \delta(x)$ is such that $y_1 \notin 1_{\mathscr{C}}$, then we have that $\varepsilon(y_1) = \varnothing$ and we can forget about the multiset $\varepsilon(y_1) \times \{y_2\}$ in the above union. Otherwise, if $y_1 \in 1_{\mathscr{C}}$, by hypothesis we must have that $y_1 = u$ and $y_2 = x'$. Moreover, the pair (u, x') occurs with multiplicity one. Therefore,

$$(\varepsilon \times \mathrm{id}_{\mathscr{C}})(\delta(x)) = \{(*, x')\}\,,$$

as desired. Now it only remains to show the same for the right-hand side triangle, but the proof is completely analogous. ■

Remark 2.3.9. As we have remarked in 2.2.10, in [RS98] the authors claim that the trivial cell-set is terminal in **Cell′**, and therefore the counit of a comonoid object in the category of pointed cell-sets is always uniquely determined. Since we have showed that this is not true (unless $\mathscr{C}$ is connected), we have chosen a valid candidate and have put the necessary conditions to ensure it is a counit. Furthermore, we want to remark that, as it has happened with proposition 2.3.3, in 2.3.8 we do not need to assume that the product δ is a pointed cell-map.

Definition 2.3.10. A cell-map $f\colon (\mathscr{C}, \delta_{\mathscr{C}}) \longrightarrow (\mathscr{D}, \delta_{\mathscr{D}})$ is called *comultiplicative* if it commutes with coproducts up to equivalence, that is, the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "f"] \arrow[d, "\delta_{\mathscr{C}}"'] & \mathscr{D} \arrow[d, "\delta_{\mathscr{D}}"] \\ \mathscr{C} \times \mathscr{C} \arrow[r, "f \times f"'] & \mathscr{D} \times \mathscr{D} \end{tikzcd}$$

commutes up to equivalence of cell-maps.

Unlike what happens with the monoid case, here we do not have that pointed comultiplicative cell-maps between comonoid objects as in 2.3.8 are automatically morphisms of comonoids, so we need to add an additional hypothesis.

Proposition 2.3.11. *Let $(\mathscr{C}, \delta_{\mathscr{C}}, \varepsilon_{\mathscr{C}})$ and $(\mathscr{D}, \delta_{\mathscr{D}}, \varepsilon_{\mathscr{D}})$ be monoid objects as in 2.3.8. Let*

$$f\colon (\mathscr{C}, \delta_{\mathscr{C}}, \varepsilon_{\mathscr{C}}) \longrightarrow (\mathscr{D}, \delta_{\mathscr{D}}, \varepsilon_{\mathscr{D}})$$

be a pointed cell-map and assume that if $x \in \mathscr{C} \setminus 1_{\mathscr{C}}$, then $f(x)$ contains no base points. Then, f is a morphism of comonoid objects in **Cell**$'$.

Proof. We have to check that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "f"] \arrow[dr, "\varepsilon_{\mathscr{C}}"'] & & \mathscr{D} \arrow[dl, "\varepsilon_{\mathscr{D}}"] \\ & \mathscr{T} & \end{tikzcd}$$

is commutative up to equivalence. In fact, we will show that it does so strictly. If $x \in 1_{\mathscr{C}}$ is a base point, then we have that

$$\varepsilon_{\mathscr{D}}(f(x)) = \{*\} = \varepsilon_{\mathscr{C}}.$$

On the other hand, if $x \in \mathscr{C} \setminus 1_{\mathscr{C}}$ is not a base point,

$$\varepsilon_{\mathscr{D}}(f(x)) = \bigcup_{y \in f(x)} \varepsilon_{\mathscr{D}}(y) = \bigcup_{y \in f(x)} \varnothing = \varnothing = \varepsilon_{\mathscr{C}}.$$

Therefore, the diagram commutes strictly. ∎

Definition 2.3.12. Let $\mathscr{C}$ be a cell-set with both a product μ and a coproduct δ. We say that μ and δ are *compatible* if the following diagram

$$\begin{tikzcd} \mathscr{C} \times \mathscr{C} \arrow[r, "\mu"] \arrow[d, "\delta \times \delta"'] & \mathscr{C} \arrow[r, "\delta"] & \mathscr{C} \times \mathscr{C} \\ \mathscr{C} \times \mathscr{C} \times \mathscr{C} \times \mathscr{C} \arrow[rr, "\mathrm{id}_{\mathscr{C}} \times \tau \times \mathrm{id}_{\mathscr{C}}"'] & & \mathscr{C} \times \mathscr{C} \times \mathscr{C} \times \mathscr{C} \arrow[u, "\mu \times \mu"'] \end{tikzcd}$$

commutes up to equivalence, which means that

$$\delta\mu \sim (\mu \times \mu)(1_{\mathscr{C}} \times \tau \times 1_{\mathscr{C}})(\delta \times \delta).$$

Observe that this condition is the same as demanding that μ is comultiplicative with respect to the coproduct structure $(\mathscr{C}, \delta)$.

Proposition 2.3.13. *Let $(C, \mu, \eta, \delta, \varepsilon)$ be a cell-set with structure maps such as in 2.3.3 and 2.3.8. Assume that μ and δ are pointed cell-maps, with μ verifying that*

$$\mu(x, y) \text{ is a base point} \iff \text{both } x \text{ and } y \text{ are base points.}$$

Suppose also that the product and the coproduct are compatible. Then, $(C, \mu, \eta, \delta, \varepsilon)$ is a bimonoid object in **Cell**$'$.

Proof. By hypothesis, we have that (C, μ, η) and (C, δ, ε) are monoid and comonoid objects in **Cell**$'$, so it only remains to check, for instance, that μ and η are comonoid morphisms. This is equivalent to check the commutativity (up to equivalence) of 4 diagrams but, since we are assuming the compatibility condition between μ and δ, the number of diagrams reduces to 3

$$\begin{tikzcd} C \otimes C \arrow[r, "\mu"] \arrow[d, "\varepsilon\otimes\varepsilon"'] & C \arrow[d, "\varepsilon"] \\ \mathscr{T} \otimes \mathscr{T} \arrow[r, "\cong"'] & \mathscr{T} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{T} \arrow[r, "\eta"] \arrow[d, "\cong"'] & C \arrow[d, "\delta"] \\ \mathscr{T} \otimes \mathscr{T} \arrow[r, "\eta\otimes\eta"'] & C \otimes C \end{tikzcd} \qquad \begin{tikzcd} \mathscr{T} \arrow[rr, "\eta"] \arrow[dr, equal] & & C \arrow[dl, "\varepsilon"] \\ & \mathscr{T}, & \end{tikzcd}$$

which trivially commute assuming the given hypothesis. ■

We conclude this section giving a sufficient condition to have a Hopf algebra in **GAb**. Recall that a Hopf algebra in **GAb**, or graded Hopf algebra, is a graded abelian group

$$H = \bigoplus_{n \geq 0} H_n$$

endowed with a product $\mu\colon H \otimes H \longrightarrow H$, a coproduct $\delta\colon H \longrightarrow H \otimes H$, a unit $\eta\colon \mathbb{Z} \longrightarrow H$, a counit $\varepsilon\colon H \longrightarrow \mathbb{Z}$ and an antipode $S\colon H \longrightarrow H$ verifying the usual axioms of Hopf algebra. Moreover, we impose that

1. $H_p H_q \subseteq H_{p+q}$,
2. $\delta(H_n) \subseteq \bigoplus_{p+q=n} H_p \otimes H_q$,
3. $S(H_n) \subseteq H_n$.

If we do not demand the existence of an antipode S, we get the definition of graded bialgebra. Furthermore, we say that H is *connected* if H_0 is zero-dimensional, that is, $H_0 \cong \mathbb{Z}$. In a graded bialgebra we can always consider the increasing filtration given by

$$H^k := \bigoplus_{i=0}^{k} H_i,$$

so that

$$H^0 \subseteq H^1 \subseteq \cdots \subseteq H^{n-1} \subseteq H^n \subseteq \dots$$

Let $(H, \mu, \delta, \eta, \varepsilon)$ be a graded connected bialgebra. Recall that the set of $\mathbb{Z}$-linear endomorphisms $\mathrm{Hom}_{\mathbb{Z}}(H, H)$ is a $\mathbb{Z}$-algebra with product given by the *convolution product*

$$\begin{array}{rccl} *\colon \mathrm{Hom}_{\mathbb{Z}}(H,H) \times \mathrm{Hom}_{\mathbb{Z}}(H,H) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(H,H) \\ (f,g) & \longmapsto & f * g := \mu \circ (f \otimes g) \circ \delta, \end{array}$$

as it is shown in the following diagram

$$H \xrightarrow{\ \delta\ } H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\ \mu\ } H.$$

The identity for this product is the map $\eta \circ \varepsilon$ which is, in general, different from the identity map id_H. The following proposition tells us that in this situation, we automatically have a Hopf algebra structure in H.

Proposition 2.3.14. *Let H be a graded connected bialgebra as before. Then, the map*

$$S = \sum_{n \geq 0} (\eta \circ \varepsilon - \mathrm{id}_H)^{*n}$$

is a well-defined $\mathbb{Z}$-linear endomorphism in $\mathrm{Hom}(H, H)$ *and acts as an antipode for H. Thus, H becomes a Hopf algebra.*

Proof. Let $R := \mathrm{Hom}(H, H)$ thought as a ring with product given by convolution. We are interested in finding the convolution inverse of $\mathrm{id}_H \in R$. In order to do so, consider the inclusion

$$i\colon R \hookrightarrow R[[X]]$$

of R into its formal power series ring. Recall that in $R[[X]]$ we have the equality

$$(1 - X)^{*-1} = \sum_{n \geq 0} X^{*n}.$$

In particular, if there is some $f \in R$ such that the expression

$$\sum_{n \geq 0} f^{*n}$$

is finite, we have that the equality

$$(1 - f)^{*-1} = \sum_{n \geq 0} f^{*n}$$

also holds in R. Observe that we can write the identity morphism id_H as

$$\mathrm{id}_H = \eta \circ \varepsilon - (\eta \circ \varepsilon - \mathrm{id}_H).$$

Thus, if we are able to show that for every $x \in H$, the expression

$$S(x) = \sum_{n \geq 0} (\eta \circ \varepsilon - \mathrm{id}_H)^{*n}(x)$$

is finite, then we are done. That said, if

$$H = \bigoplus_{n \geq 0} H_i$$

is a graded connected bialgebra, we have an increasing filtration

$$H^0 \subseteq H^1 \subseteq \cdots \subseteq H^{n-1} \subseteq H^n \subseteq \ldots,$$

with

$$H^k = \bigoplus_{i=0}^{k} H_i$$

and $H^0 = H_0 \cong \mathbb{Z}$. In particular, we have that $\eta \circ \varepsilon | H^0 = \mathrm{id}_{H^0}$. From this fact, we will show by induction that

$$H^{n-1} \subseteq \ker(\eta \circ \varepsilon - \mathrm{id}_H)^{*n},$$

for every $n \geq 1$. The base case $n = 1$ is the above-mentioned condition. Suppose that $n > 1$ and let $H^{n-1} \subseteq \ker(\eta \circ \varepsilon - \mathrm{id}_H)^{*n}$ be the induction hypothesis. Then,

$$(\eta \circ \varepsilon - \mathrm{id}_H)^{*(n+1)} = \mu^n \circ (\eta \circ \varepsilon - \mathrm{id}_H)^{\otimes(n+1)} \circ \delta^n.$$

Now observe that, since

$$\delta(H_n) \subseteq \bigoplus_{p+q=n} H_p \otimes H_q,$$

we can use our induction hypothesis to obtain that

$$H^n \subseteq \ker(\eta \circ \varepsilon - \mathrm{id}_H)^{*(n+1)}.$$

For every $x \in H$, let

$$n_x := \min \{ n \in \mathbb{N} : x \in H^n \}.$$

Then,

$$S(x) = \sum_{i=0}^{n_x} (\eta \circ \varepsilon - \mathrm{id}_H)^{*i}$$

is a finite sum, which gives a well defined convolution inverse for 1_H. ∎

Thus, at the level of cell-sets, we conclude the following.

Corollary 2.3.15. *Let $(\mathscr{C}, \mu, \eta, \delta, \varepsilon)$ be a bimonoid object in* **Cell′** *as in 2.3.13. Assume that $\mathscr{C}$ is a connected cell-set. Then, the free abelian group $Z_*(\mathscr{C})$ is a graded Hopf algebra.*

Cell-sets as in 2.3.15 are termed *Hopf cell-sets*. We now give some fundamental examples of such a class of cell-sets.

Example 2.3.16. *Let $\mathscr{S}$ be the family of all finite sets. Define an equivalence relation on $\mathscr{S}$ by*

$$U \sim V \iff |U| = |V|,$$

and, for every $U \in \mathscr{S}$, take $d(U) = |U|$ as the dimension function. Then, each $\mathscr{S}_n$ is formed by exactly one equivalence class. In particular, $\mathscr{S}$ is a connected cell-set with base point the empty set $\emptyset$, and the free abelian group $Z_*(\mathscr{S})$ has one generator x_k in each dimension $k \in \mathbb{Z}$. We can endow $\mathscr{S}$ with a product defined as

$$\begin{array}{rcl} \mu\colon \mathscr{S} \times \mathscr{S} & \longrightarrow & \mathscr{S} \\ (U,V) & \longmapsto & \mu(U,V) := U \amalg V, \end{array}$$

and clearly the induced product in $Z_*(\mathscr{S})$ is

$$\begin{array}{rcl} \mu_*\colon Z_*(\mathscr{S}) \otimes Z_*(\mathscr{S}) & \longrightarrow & Z_*(\mathscr{S}) \\ x_k \otimes x_l & \longmapsto & \mu_*(x_k \otimes x_l) = x_{k+l}. \end{array}$$

We can also define a coproduct via

$$\begin{array}{rcl} \delta\colon \mathscr{S} & \longrightarrow & \mathscr{S} \times \mathscr{S} \\ U & \longmapsto & \delta(U) := \{(V,W) : V \amalg W = U\}\,. \end{array}$$

Let us compute the induced coproduct in $Z_*(\mathscr{S})$. For every $k \in \mathbb{Z}$, write $x_k = [U_k]$. Observe that

$$\delta(U_k) = \coprod_{i=0}^{k} \delta_i(U_k),$$

where

$$\delta_i(U) := \{(V,W) \in \delta(U_k) : |V| = i,\ |W| = k-i\}\,.$$

It is clear that, for every $0 \le i \le k$, $|\delta_i(U)| = \binom{k}{i}$. Therefore, the induced coproduct is given by

$$\begin{array}{rcl} \delta_*\colon Z_*(\mathscr{S}) & \longrightarrow & Z_*(\mathscr{S}) \otimes Z_*(\mathscr{S}) \\ x_k & \longmapsto & \delta_*(x_k) = \sum_{i=0}^{k} \binom{k}{i} x_i \otimes x_{k-i}. \end{array}$$

The conditions in 2.3.15 are satisfied, so that $\mathscr{S}$ is a Hopf cell-set, with Hopf algebra $Z_*(\mathscr{S})$ isomorphic to the polynomial algebra $\mathbb{Z}[X]$, with coproduct given by $\delta_*(X) = X \otimes 1 + 1 \otimes X$ and antipode equal to $S(X) = -X$.

Example 2.3.17. *A partition is a finite set σ consisting of a collection of non-empty finite sets, called blocks, which are pairwise disjoint. If there is a set V that coincides with the union of the blocks of σ, we say that σ is a partition of V. Let $\mathscr{P}_\pi$ be the collection of all partitions. We define an equivalence relation on $\mathscr{P}_\pi$ as follows: given $\sigma, \tau \in \mathscr{P}_\pi$, we declare them to be equivalent if there exists a bijection $f\colon \sigma \longrightarrow \tau$ that preserves cardinalities. For every $\sigma \in \mathscr{P}_\pi$, we set*

$$\begin{array}{rcl} d\colon \mathscr{P}_\pi & \longrightarrow & \mathbb{Z} \\ \sigma & \longmapsto & d(\sigma) := \sum_{B \in \sigma} |B|. \end{array}$$

This map is clearly a dimension function by construction of the equivalence relation. Thus $\mathscr{P}_\pi$ is a cell-set, which is clearly connected with base point the empty partition $\varnothing$. Now we are going to add more structure. We can define a product by letting

$$\begin{array}{rcl} \mu\colon \mathscr{P}_\pi \times \mathscr{P}_\pi & \longrightarrow & \mathscr{P}_\pi \\ (\sigma,\tau) & \longmapsto & \mu(\sigma,\tau) := \sigma \amalg \tau. \end{array}$$

For the coproduct, let us first introduce some notation. If σ is a partition of the set V and $U \subseteq V$ is a subset of V, we define the restriction $\sigma|U$ of σ to the subset U as the partition of U given by

$$\sigma|U := \{B \cap U : B \in \sigma,\ B \cap U \neq \varnothing\}.$$

We can now define a coproduct on $\mathscr{P}_\pi$ via the map

$$\begin{array}{rcl} \delta\colon \mathscr{P}_\pi & \longrightarrow & \mathscr{P}_\pi \times \mathscr{P}_\pi \\ \sigma & \longmapsto & \delta(\sigma), \end{array}$$

where, for every $\sigma \in \mathscr{P}_\pi$ such that σ is a partition of the set V,

$$\delta(\sigma) := \{(\sigma|U, \sigma|W) : U \amalg W = V\}.$$

It can be seen that this cell-set is in fact a Hopf cell-set, isomorphic to a polynomial algebra (the details can be found in [RS98], 221).

The following example is a generalization of the previous two.

Example 2.3.18. *Let $G = (V(G), E(G))$ be a graph. Recall that if $U \subseteq V(G)$ is a subset of the vertices of G, then the induced subgraph $G|U$ is the graph defined as*

$$G|U := (U, E(G|U)),$$

where $E(G|U) = E(G) \cap \binom{U}{2}$. Let $\mathscr{G}$ be a subclass of the class of all finite graphs which is closed under disjoint unions and induced subgraphs. We can turn $\mathscr{G}$ into a cell-set by considering equivalence given by graph isomorphism, and letting

$$\begin{array}{rcl} d\colon \mathscr{G} & \longrightarrow & \mathbb{Z} \\ G & \longmapsto & d(G) := |E(G)|. \end{array}$$

We can define a product on $\mathscr{G}$ given by disjoint union of graphs

$$\begin{array}{rcl} \mu\colon \mathscr{G} \times \mathscr{G} & \longrightarrow & \mathscr{G} \\ (G, H) & \longmapsto & \mu(G, H) := G \amalg H = (V(G) \amalg V(H), E(G) \amalg E(H)), \end{array}$$

and a coproduct

$$\begin{array}{rcl} \delta\colon \mathscr{G} & \longrightarrow & \mathscr{G} \times \mathscr{G} \\ G & \longmapsto & \delta(G) \end{array}$$

by setting

$$\delta(G) := \{(G|U, G|W) : U \amalg W = V(G)\}.$$

The Hopf algebra $Z_*(\mathscr{G})$ is also polynomial, with an even more complicated structure that can be found in [RS98], 222.

2.4 Partially Ordered Sets

The following two sections of this chapter are devoted to explain an important class of cell-sets coming from posets. We do not go deeper into such a class of examples since, for lack of time, we cannot express such examples in the language of decomposition spaces. Of course, the detailed explanation can be found in [RS98].

A *partially ordered set* (or *poset*, for short) is a pair $(P, \leq)$, where P is a set and $\leq$ is a binary relation on P which is reflexive, antisymmetric and transitive. As usual, we will denote a poset only with its ground set, whenever the order relation is understood or not relevant. If necessary, we will write $\leq_P$ for the order relation of a poset with underlying set P.

Now we are going to give several definitions concerning posets. In order to avoid invoking a different poset every time we give a new definition, let us fix an arbitrary poset P. A *weak subposet* of P is a subset $Q \subseteq P$ of P which is partially ordered in such a way that if $x \leq_Q y$, then $x \leq_P y$ for every $x, y \in Q$. If this implication turns out to be an equivalence, that is, if

$$x \leq_Q y \iff x \leq_P y, \text{ for every } x, y \in Q,$$

then we say that Q is an *induced subposet*. By a subposet of P we will always mean an induced subposet. We also say that a subposet of P is *spanning* if its sets of minimal and maximal elements are respectively contained in the sets of minimal and maximal elements of P. Given a pair of elements $x, y \in P$, the *subinterval*

$$[x, y] := \{z \in P : x \leq z \leq y\}$$

is a subposet of P. The poset P is said to have a $\widetilde{0}$ or a $\widetilde{1}$ if it has a minimum or maximum element, respectively. If that is the case, then such a poset is equal to the interval $P = [\widetilde{0}, \widetilde{1}]$ and, for every $z \in P$, we set $P_z := [\widetilde{0}, z]$ and $P^z := [z, \widetilde{1}]$, which are of course well-defined intervals.

If $x, y \in P$ are elements of our poset P, we say that y *covers* x if $x < y$ and there is no $z \in P$ such that $x < z < y$. Recall that a *chain* is a poset C in which every element is comparable, that is, if $x, y \in C$, then either $x \leq y$ or $y \leq x$; and a chain of the poset P is just a subposet of P which is a chain. There are two interesting notions of chain. A chain in P is *maximal* if no element can be added to it without losing the property of being totally ordered. A slightly different and weaker version is the following one: a *saturated chain* in P is a chain such that no element can be added *between two of its elements* without losing the property of being totally ordered. Observe that being maximal is stronger than being saturated, since it also excludes the possibility of adding elements either less than all elements of the chain or greater than all its elements. If the chain $C = \{x_0 < x_1 < \cdots < x_n\}$ is finite, then it is saturated if and only if for every $1 \leq i \leq n$, x_i covers x_{i-1}. A finite saturated chain is maximal if and only if it is spanning. Given a finite chain C, we define its length $l(C)$ to be $l(C) = |C| - 1$. The length of an interval $[x, y]$ is usually denoted by $l(x, y)$.

In this final part of definitions, let us assume that P is finite. The length of P is given by

$$l(P) := \max\{l(C) : C \text{ is a chain of } P\}.$$

If every maximal chain of P has the same length, we say that P is *graded of rank* n. A *subgrading* on P is a function $\rho\colon P \longrightarrow \mathbb{N}$ such that, for every $x, y \in P$, if $x <_P y$, then $\rho(x) < \rho(y)$. We say that P is *subgraded* if it is equipped with a subgrading ρ, and therefore every subposet $Q \subseteq P$ of P is subgraded just by restriction $\rho|Q$ of the subgrading. If P is graded, then there is a canonical subgrading determined by the formula

$$\begin{aligned} \rho\colon P &\longrightarrow \mathbb{N} \\ x &\longmapsto \rho(x) := \begin{cases} 0, \text{ if } x \text{ is minimal,} \\ \rho(z)+1, \text{ if } z \text{ covers } x. \end{cases} \end{aligned}$$

In particular, $l(x, y) = \rho(y) - \rho(x)$. We refer to this canonical subgrading as the *rank function* of P. Unless otherwise stated, we will always assume that a graded poset comes equipped with such a rank function.

2.5 Interval Cell-Sets and Categories

Definition 2.5.1. A cell set $(\mathscr{C}, \sim, d)$ is said to be a *subgraded interval cell-set* if $\mathscr{C}$ consists of finite subgraded intervals and, for every $I = [a, b] \in \mathscr{C}$,

$$d(I) = \rho_I(b) - \rho_I(a), \tag{2.6}$$

where $\rho_I\colon I \longrightarrow \mathbb{N}$ is the subgrading of I. In particular, it follows that for every $I \in \mathscr{C}$,

$$\begin{cases} d(I) \geq 0, \\ d(I) = 0 \iff |I| = 1, \\ d(I) = d(I_x) + d(I^x), \end{cases}$$

where the last property holds for all $x \in I$ such that $I_x, I^x \in \mathscr{C}$. If $\mathscr{C}$ only consists of graded intervals (with, of course, their respective canonical gradings), we say that $(\mathscr{C}, \sim, d)$ is a *graded interval cell-set.* If we want to make common reference to both kinds of cell-sets, or if there is no confusion by the context, we refer to both terms as *interval cell-set.* Given an interval cell-set $\mathscr{C}$, we say that it is closed if it is closed under formation of subintervals and its equivalence relation is *order compatible*, meaning that if $I \sim J$ in $\mathscr{C}$, then there exists a bijection $f\colon I \longrightarrow J$ such that

$$I_x \sim J_{f(x)} \text{ and } I^x \sim J^{f(x)}$$

for every $x \in I$.

Proposition 2.5.2. *Let $\mathscr{C}$ be a closed interval cell-set. Then, the map given by*

$$\begin{aligned} \delta\colon \mathscr{C} &\longrightarrow \mathscr{P}(\mathscr{C} \times \mathscr{C}) \\ I &\longmapsto \delta(I) := \{(I_x, I^x) : x \in I\} \end{aligned}$$

induces a coproduct on $\mathscr{C}$.

The cell-map in the above proposition is termed the *incidence coproduct*, and the coalgebra $Z_*(\mathscr{C})$ is named the *incidence coalgebra* of $\mathscr{C}$. The induced coproduct on $Z_*(\mathscr{C})$ is obviously determined by

$$\delta_*([I]) = \sum_{x \in I} [I_x] \otimes [I^x].$$

We also have a counit given by

$$\varepsilon_*([I]) = \begin{cases} 1, & \text{if } |P| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

which is induced by the cell map in 2.3.8 whenever $\mathscr{C}$ is pointed.

In practice, the majority of interval cell-sets that appear occur as object cell-sets of certain cell-categories of intervals.

Definition 2.5.3. We define **Int** to be the category of finite intervals of a given poset and order preserving maps. By $\mathbf{I_g}$ we mean the full subcategory of finite graded intervals, and $\mathbf{I_s}$ makes reference to the category of finite subgraded intervals and *dimension preserving* maps $f\colon I \longrightarrow J$ (which are also required to be order preserving) that verify

$$\rho_I(y) - \rho_I(x) = \rho_J(f(y)) - \rho_J(f(x)),$$

for every $x \leq y$ in I.

Let $f\colon I \longrightarrow J$ be a morphism in **Int**, and write $\hat{f}$ for the unique map that makes the following diagram commutative

$$\begin{tikzcd} I \arrow[rr, "f"] \arrow[dr, "\hat{f}"'] & & J \\ & f(P). \arrow[ur, hook, "i_{f(P),Q}"'] & \end{tikzcd}$$

We have then a canonical factorization

$$f = i_{f(P),Q} \circ \hat{f} \tag{2.7}$$

within the category **Int**.

Definition 2.5.4. A *subgraded interval category* is a subcategory $\widetilde{\mathbf{I}}_s$ of $\mathbf{I}_s$ with the property that every morphism preserves the canonical factorization 2.7, and which is equipped with the dimension function 2.6. Analogously, a *graded interval category* is a subcategory $\widetilde{\mathbf{I}}_g$ of $\mathbf{I}_g$ with the same properties. As before, we refer to both terms as *interval category* whenever there is no danger of confusion, or if there is no need of distinction. Observe that subgraded and graded interval categories are cell-categories, whose respective object cell-sets are interval cell-sets (that is, both graded and subgraded) and graded interval cell-sets.

Let $\mathcal{C}$ be an interval category. We say that $\mathcal{C}$ is *closed* if for every object I and every subinterval $R \subseteq I$ of I, the inclusion $i_{R,I} \colon R \hookrightarrow I$ is a morphism in $\mathcal{C}$. Of course, $\mathbf{I}_g$ and $\mathbf{I}_s$ are both closed interval categories. More terminology: if $\mathcal{C}$ is a full subcategory of either $\mathbf{I}_s$ or $\mathbf{I}_g$, then obviously every morphism of $\mathcal{C}$ retains its canonical factorization. In this case, we refer to $\mathcal{C}$ as a *full interval category.* If $\mathcal{C}$ is a full interval category, then its associated interval cell-set must be connected, and $\mathcal{C}$ is closed if and only if it is closed under the formation of subintervals.

Proposition 2.5.5. *Let $\mathcal{C}$ be a closed interval category. Then, its object cell-set is a closed interval cell-set.*

Proof. Assume that $\mathcal{C}$ is a closed interval category and denote by $\mathscr{C}$ its associated object cell-set, which is already an interval cell-set. Let $I \in \mathscr{C}$ be an interval and let $R \subseteq I$ be a subinterval of I. Then, since the inclusion $i_{R,I}$ lies within $\mathcal{C}$, in particular we have that R also belongs to $\mathscr{C}$. Therefore, $\mathscr{C}$ is closed under formation of subintervals. Consider now a pair of intervals $I, J \in \mathscr{C}$ such that $I \sim J$, that is, there exists an isomorphism $f \colon I \longrightarrow J$ in $\mathcal{C}$. The restriction

$$f|R = f \circ i_{R,I}$$

belongs to $\mathcal{C}$ because it is the composition of maps that are in $\mathcal{C}$. Hence, since in an interval category every morphism retains its canonical factorization, the map

$$\widehat{f|R} \colon R \longrightarrow f(R)$$

is a morphism in $\mathcal{C}$, which is indeed an isomorphism. In particular, letting R equal to I_x and I^x, with $x \in I$, we deduce that

$$I_x \sim Q_{f(x)} \text{ and } I^x \sim Q^{f(x)},$$

for every $x \in I$. Thus, $\mathscr{C}$ is a closed interval cell-set. ∎

Example 2.5.6. *Let $\mathbf{L}$ denote the full, closed graded interval category defined by all finite non-empty chains as objects, with dimension function given by the length of the chains. In particular this means that morphisms are just order preserving maps. Let $\mathscr{L}$ denote its associated object cell-set, which is a closed interval category by proposition 2.5.5. The free abelian group $Z_*(\mathscr{L})$ has one and only one generator β_k in each non-negative dimension $k \geq 0$, with canonical representative given by the chain $[k+1] = \{1 < \cdots < k+1\}$. Since $\mathscr{L}$ is a closed interval cell-set, it has incidence coproduct given by 2.5.2, which induces the divided powers coproduct*

$$\delta_*(\beta_k) = \sum_{i=0}^{k} \beta_i \otimes \beta_{k-i}.$$

Its counit is given by

$$\varepsilon_*(\beta_k) = \begin{cases} 1, & \textit{if } k = 0, \\ 0, & \textit{otherwise,} \end{cases}$$

which, since $\mathscr{L}$ is connected, it is induced by the cell-map in 2.3.8.

For the next example, we need a definition. A *composition* of a non-negative integer n is a sequence of positive integers $a = (a_1, \dots, a_k)$ such that $a_1 + \cdots + a_k = n$. By convention, the integer 0 has only one composition given by $a = 0$.

Example 2.5.7. *Let $\mathbf{L}^+$ be the full, closed subgraded interval category of all finite subgraded chains and dimension preserving maps. If $L = \{x_0 < \cdots < x_k\}$ is a subgraded chain, we write $a(L) = (a_1, \dots, a_k)$ for the composition of $d(L)$ given by $a_i = d([x_{i-1}, x_i])$, for every $1 \leq i \leq k$. Let $\mathscr{L}^+$ be its corresponding object cell-set. For each $n \geq 0$, we claim that the free abelian group $Z_*(\mathscr{L}^+)$ has as many generators β_a as compositions a of n, where every β_a is represented by any subgraded chain L with $a(L) = a$. That is, we want to check that*

$$\mathscr{L}^+_n / \sim = \{\beta_a : a \in c(n)\},$$

where $c(n)$ is the set of compositions of n. First, it is clear that, for every composition a of n, every representative of β_a gives an element of dimension n, just by construction. Conversely, let $L = \{x_0 < \cdots < x_k\}$ be a subgraded chain with $d(L) = n$. Then, it is clear that $a(L)$ is a composition of n, and therefore L belongs to $\beta_{a(L)}$. Finally, it just remains to see that if $a \neq a'$ are two different compositions of n, then $\beta_a \cap \beta_{a'} = \varnothing$. Let $L = \{x_0 < \cdots < x_k\}$ and $L' = \{y_0 < \cdots < y_l\}$ be representatives of β_a and $\beta_{a'}$, respectively. If $k \neq l$ we are done, so we can assume that $k = l$. If the compositions are different, this means that there exists some $i \in \{1, \dots, k\}$ such that $d([x_{i-1}, x_i]) \neq d([y_{i-1}, y_i])$, but this implies that there is no bijective dimension preserving map between L and L', so they belong to different equivalence classes.

Again, by 2.5.5, the cell-set $\mathscr{L}^+$ is closed and, hence, proposition 2.5.2 ensures that there is a coproduct on $Z_*(\mathscr{L}^+)$ given by

$$\delta_*(\beta_a) = \sum_{c * d = a} \beta_c \otimes \beta_d,$$

where the operation $*$ means the concatenation of compositions, defined in the natural way. For the counit, observe that the zero-dimensional subgraded chains are those who have only one element, and all of them are related since given a pair of subgraded singletons, the unique bijective map between them is dimension preserving by construction. In particular $\mathscr{L}^+$ is connected, and therefore the cell-map in proposition 2.3.8 induces the counit

$$\varepsilon_*(\beta_a) = \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Chapter 3

The Cellularization Functor

In this chapter, we take the first step towards the desired relationship between cell-sets and decomposition spaces. Concretely, we construct a full and bijective on objects functor from the category of spans of discrete groupoids with some additional conditions to **Cell**. Moreover, we equip this modified span category with a symmetric monoidal structure such that this functor becomes a braided strict monoidal functor, and therefore it preserves the derived objects from the monoidal structure.

3.1 From Cell-Sets to Discrete Groupoids

We first review some basic notions related to discrete groupoids, we construct the functor on objects and we check that it is indeed a bijection.

Definition 3.1.1. We say that a small category $\mathcal{C}$ is *discrete* if for every pair of objects $x, y \in \mathcal{C}$, there exists at most one arrow between them, that is,

$$|\operatorname{Hom}_{\mathcal{C}}(x, y)| \leq 1.$$

Sets are discrete categories with only identity arrows: given any set X, we can construct a discrete category whose set of objects is X and its set of arrows consists only of identities. Conversely, a small category with only identity arrows defines a set. Thus sets are a particular case of discrete categories. Given a set X, we refer to its associated category as the *discrete category on* X.

Functors between discrete categories are easy to define: the existence of at most one arrow between every pair of objects implies that functoriality is automatically fulfilled.

Proposition 3.1.2. *Let* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ *be an assignment between discrete categories defined as follows: a map between the sets of objects*

$$F\colon \operatorname{Obj}(\mathcal{C}) \longrightarrow \operatorname{Obj}(\mathcal{D})$$

and, for every pair of objects $x, y \in \mathcal{C}$*, a map between the hom-sets*

$$F = F_{x,y}\colon \operatorname{Hom}_{\mathcal{C}}(x, y) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(F(x), F(y)).$$

Then, without further conditions, F *is a functor.*

Proof. For every $x \in \mathscr{C}$, the identity arrow at x must be mapped to an arrow in $\mathrm{Hom}_{\mathcal{D}}(F(x), F(x))$. Since $\mathcal{D}$ is discrete,

$$\mathrm{Hom}_{\mathcal{D}}(F(x), F(x)) = \left\{\mathrm{id}_{F(x)}\right\},$$

and therefore $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$. Given now a pair of composable arrows

$$x \xrightarrow{g} y \xrightarrow{f} z,$$

both $F(f \circ g)$ and $F(f) \circ F(g)$ belong to $\mathrm{Hom}_{\mathcal{D}}(F(x), F(z))$

$$\begin{tikzcd} F(x) \arrow[r, "F(g)"] \arrow[rr, bend right, "F(f\circ g)"'] & F(y) \arrow[r, "F(f)"] & F(z). \end{tikzcd}$$

Since $\mathcal{D}$ is discrete, we must have that $F(f \circ g) = F(f) \circ F(g)$. ■

Another interesting property is that functors between discrete categories behave like maps between sets: if we want to see that two functors between discrete categories are equal, it suffices to check that they agree on objects.

Proposition 3.1.3. *Let $F, G \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a pair of functors between discrete categories. Then,*

$$F = G \iff F(x) = G(x), \textit{ for every } x \in \mathrm{Obj}(\mathcal{C}).$$

Proof. Suppose that F and G agree on objects. Then, for every arrow $f \colon x \longrightarrow y$ in $\mathcal{C}$, we have that

$$\begin{tikzcd} F(x) \arrow[r, bend left, "F(f)"] \arrow[d, equal] & F(y) \arrow[d, equal] \\ G(x) \arrow[r, bend right, "G(f)"'] & G(y). \end{tikzcd}$$

Hence, by uniqueness, we must have that $F(f) = G(f)$. ■

This property is useful when dealing with complicated diagrams. If all the involved arrows are functors between discrete groupoids, then it suffices to check commutativity on objects. We may use this fact throughout the work without mention.

We now focus on a specific kind of discrete categories whose role is of major importance in our work.

Definition 3.1.4. Let **Grpd** be the category of small groupoids and functors between them. We define **DiscGrpd** as the full subcategory of **Grpd** consisting of discrete groupoids.

Regarding the set of integers $\mathbb{Z}$ as a discrete category, we can consider the slice category $\mathbf{DiscGrpd}/\mathbb{Z}$. Recall that its objects are functors

$$D_G \colon G \longrightarrow \mathbb{Z},$$

where G is a discrete groupoid, and its arrows are commutative triangles of the form

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "D_G"'] & & H \arrow[dl, "D_H"] \\ & \mathbb{Z}, & \end{tikzcd}$$

where f is a functor. Composition in $\mathbf{DiscGrpd}/\mathbb{Z}$ is the same as in $\mathbf{DiscGrpd}$ and identities are also the same. We refer to the objects of $\mathbf{DiscGrpd}/\mathbb{Z}$ as *discrete groupoids over* $\mathbb{Z}$.

It is well-known that preorders can be identified with discrete categories. In the same way, sets with an equivalence relation are naturally in one-to-one correspondence with discrete groupoids: the symmetry property allows us to invert the arrows. With this philosophy, we identify discrete groupoids over the integers with cell-sets. In order to avoid technical issues, we define the following equivalence relation on $\mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z})$:

$$\begin{array}{c} C \\ \big\downarrow D_C \\ \mathbb{Z} \end{array} \sim \begin{array}{c} C' \\ \big\downarrow D_{C'} \\ \mathbb{Z} \end{array} \iff \begin{array}{c} \mathrm{Obj}(C) = \mathrm{Obj}(C'),\ |\operatorname{Hom}_C(x,y)| = |\operatorname{Hom}_{C'}(x,y)| \\ \text{for every } x, y \in \mathrm{Obj}(C) = \mathrm{Obj}(C') \text{ and} \\ D_C|_{\mathrm{Obj}(C)} = D_{C'}|_{\mathrm{Obj}(C')}. \end{array}$$

Then, we have the desired result.

Proposition 3.1.5. *There is a one-to-one correspondence between cell-sets and discrete groupoids over $\mathbb{Z}$ modulo the above equivalence relation, that is, there exists a bijective map*

$$\widetilde{\Psi} \colon \mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z})/\sim \longrightarrow \mathrm{Obj}(\mathbf{Cell}).$$

Proof. We first construct a map

$$\mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z}) \xrightarrow{\Psi} \mathrm{Obj}(\mathbf{Cell})$$

as follows: let $D_G \colon G \longrightarrow \mathbb{Z}$ be a discrete groupoid over $\mathbb{Z}$. Then, $\Psi(G)$ is the cell-set that has $\mathrm{Obj}(G)$ as underlying set, equivalence relation given by

$$x \sim_{\Psi(G)} y \iff |\operatorname{Hom}_G(x,y)| = 1$$

and dimension function determined by the functor D_G acting on objects: it defines a map

$$d = d_{\Psi(G)} \colon \mathrm{Obj}(G) \longrightarrow \mathbb{Z}$$

which is constant on equivalence classes: if $x \sim y$, then the functor D_G must map the unique arrow $x \longrightarrow y$ to the identity arrow on $d(x) = d(y)$

$$\begin{tikzcd} x \arrow[r] & y \arrow[r, mapsto, "D_G"] & d(x) = d(y) \arrow[loop above, "\mathrm{id}_{d(x)=d(y)}"] \end{tikzcd}.$$

Using the definition we have given for Ψ, it is straightforward to check that

$$\begin{array}{c} C \\ \downarrow{\scriptstyle D_C} \\ \mathbb{Z} \end{array} \sim \begin{array}{c} C' \\ \downarrow{\scriptstyle D_{C'}} \\ \mathbb{Z} \end{array} \iff \Psi(C) = \Psi(C').$$

Therefore, Ψ factors uniquely through the quotient $\mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z}) /\!\sim$

$$\begin{tikzcd} \mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z}) \arrow[r, "\Psi"] \arrow[d, two heads] & \mathrm{Obj}(\mathbf{Cell}) \\ \mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z})/\sim . \arrow[ur, dashed, "\widetilde{\Psi}"'] & \end{tikzcd}$$

Let us now construct the inverse of $\widetilde{\Psi}$. As before, we first construct a map

$$\mathrm{Obj}(\mathbf{Cell}) \xrightarrow{\Psi^{-1}} \mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z}).$$

We want to remark that Ψ^{-1} is not the inverse of Ψ, but its composition with the projection will be the inverse of $\widetilde{\Psi}$, so the notation is justified. That said, let $(\mathscr{C}, \sim, d)$ be a cell-set. Then, $\Psi^{-1}(\mathscr{C})$ is the discete groupoid defined as follows: take $\mathscr{C}$ as the set of objects and for every $x, y \in \mathscr{C}$, we declare that $\mathrm{Hom}_{\Psi^{-1}(\mathscr{C})}(x, y)$ consists of exactly one arrow if $x \sim y$ and it equals the empty set otherwise. Since the binary relation defined on $\mathscr{C}$ is an equivalence relation, the axioms of groupoid trivially hold:

1. **Identity arrows.** Let $x \in \mathscr{C}$. Since $\sim$ is reflexive, by construction there exists a unique morphism $x \longrightarrow x$. We define this morphism as the identity arrow on x. By the uniqueness property of the hom-sets, it is easy to see that such an arrow plays, indeed, the role of the identity.

2. **The arrows are invertible**. Let $x, y \in \mathscr{C}$ such that there exists one arrow between them, that is, $x \sim y$. Since $\sim$ is symmetric, we deduce that there exists a unique morphism $y \longrightarrow x$. We define this morphism as the inverse of the arrow $x \longrightarrow y$. Again, the construction of the hom-sets makes this definition consistent: compositions on both sides must be the corresponding identities by uniqueness.

3. **The arrows are composable**. Given $x \longrightarrow y$ and $y \longrightarrow z$, by transitivity we have that there exists an unique arrow $x \longrightarrow z$. We define this arrow as the required composition. Once again, by uniqueness, the associativity property also holds.

Thus, we have constructed a discrete groupoid $\Psi^{-1}(\mathscr{C})$. It only remains to define the dimension functor $D_{\Psi^{-1}(\mathscr{C})}\colon \Psi^{-1}(\mathscr{C}) \longrightarrow \mathbb{Z}$. In objects, we define it as the dimension function $d\colon \mathscr{C} \longrightarrow \mathbb{Z}$ associated to the cell-set $\mathscr{C}$. In arrows, the only possibility is to send each arrow $x \longrightarrow y$ to the identity on $d(x) = d(y)$, which works precisely because d is constant on equivalence classes. This construction yields a well-defined map

$$\mathrm{Obj}(\mathbf{Cell}) \xrightarrow{\Psi^{-1}} \mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z}),$$

whose composition with the projection onto the quotient

$$\begin{tikzcd}
\mathrm{Obj}(\mathbf{Cell}) \arrow[r, "\Psi^{-1}"] \arrow[dr, dashed, "\widetilde{\Psi^{-1}}"'] & \mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z}) \arrow[d, two heads] \\
& \mathrm{Obj}(\mathbf{DiscGrpd}/\mathbb{Z})/\sim
\end{tikzcd}$$

gives the inverse of $\widetilde{\Psi}$, as one can easily check. ■

Remark 3.1.6. In practice, in order to avoid needless notation, we will identify the maps $\Psi = \widetilde{\Psi}$ and $\Psi^{-1} = \widetilde{\Psi^{-1}}$. We could have even worked without using the quotient and simply declaring equivalent discrete groupoids to be equal, but we find this discussion more enlightening.

3.2 The Cellularization Functor

In the previous section we have seen that giving a cell-set is the same as giving a discrete groupoid over $\mathbb{Z}$. This fact naturally suggests the existence of a functor relating both categories. Because of the structure of cell-maps, it is not enough to stay in the slice category, so we will go a step further considering spans of discrete groupoids over $\mathbb{Z}$.

3.2.1 Bicategory of Spans

Before getting into the construction of the functor, we need to define the right framework to talk about spans, and this is done using the bicategory of spans. Let us first give a general definition.

Definition 3.2.1. Let $\mathcal{C}$ be a category with pullbacks. For every cospan in $\mathcal{C}$

$$A \xrightarrow{f} B \xleftarrow{g} C,$$

we choose a pullback diagram

$$\begin{tikzcd}
A \times_B C \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

using the axiom of choice. Recall that a span in $\mathcal{C}$ is a diagram of the form

$$A \xleftarrow{f} B \xrightarrow{g} C.$$

We will sometimes refer to it as (f, g) whenever its whole structure is understood, and analogously with cospans. With this procedure we can construct the *bicategory of spans in* $\mathcal{C}$, denoted by $\mathbf{Span}(\mathcal{C})$, consisting of the following data:

- Its objects, or 0-cells, are the same as in $\mathcal{C}$.
- For every pair of objects $A, C \in \mathcal{C}$, there is a category $\mathbf{Span}(A, C)$ whose objects, or 1-cells, are the spans in $\mathcal{C}$ from A to C. For every $A \in \mathcal{C}$, its identity arrow in $\mathbf{Span}(A, A)$ consists of two copies of the identity arrow of A in $\mathcal{C}$, that is,
$$A = A = A.$$
- A morphism, or 2-cell, in the category $\mathbf{Span}(A, C)$ between two spans from A to C
$$(A \xleftarrow{f} B \xrightarrow{g} C) \xrightarrow{\varphi} (A \xleftarrow{f'} B' \xrightarrow{g'} C)$$
is a morphism $\varphi\colon B \longrightarrow B'$ in $\mathcal{C}$ such that the diagram
$$\begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\varphi"] & \\ A & & C \\ & B' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$
is commutative. It is clear from the definition that identity 2-cells are just the identity morphisms in $\mathcal{C}$, and vertical composition is the composition in $\mathcal{C}$.
- Here comes the interesting part: horizontal composition of 1-cells is induced by the chosen pullbacks. This means the following: if we have a span from A to B and a span from B to C
$$\begin{tikzcd} & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ A & & B & & C, \end{tikzcd}$$
we can take the chosen pullback of the cospan in the middle (f_2, g_1)
$$\begin{tikzcd} & & M \times_B N \arrow[dl, "\widetilde{g_1}"'] \arrow[dr, "\widetilde{f_2}"] \arrow[ddll, bend right, "f_1 \circ \widetilde{g_1}"'] \arrow[ddrr, bend left, "g_2 \circ \widetilde{f_2}"] & & \\ & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ A & & B & & C \end{tikzcd}$$
and hence the horizontal composition $(g_1, g_2) \circ (f_1, f_2)$ is given by the span $(f_1 \circ \widetilde{g_1}, g_2 \circ \widetilde{f_2})$ from A to C.

- The horizontal composition of 2-cells is induced by the universal property of pullbacks. Let φ_1 and φ_2 be a pair of horizontally composable 2-cells as in the following diagram.

$$\begin{tikzcd}
 & & M \times_B N \arrow[dl, "\widetilde{g}_1"'] \arrow[dr, "\widetilde{f}_2"] \arrow[ddll, bend right, "f_1 \circ \widetilde{g}_1"'] \arrow[ddrr, bend left, "g_2 \circ \widetilde{f}_2"] & & \\
 & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\varphi_1"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] \arrow[dd, "\varphi_2"] & \\
A & & B & & C. \\
 & M' \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] & & N' \arrow[ul, "g_1'"'] \arrow[ur, "g_2'"] & \\
 & & M' \times_B N' \arrow[ul, "\widetilde{g}_1'"] \arrow[ur, "\widetilde{f}_2'"'] \arrow[uull, bend left, "f_1' \circ \widetilde{g}_1'"] \arrow[uurr, bend right, "g_2' \circ \widetilde{f}_2'"'] & &
\end{tikzcd}$$

It easily follows from the universal property of the bottom pullback that there exists a unique morphism ψ

$$\begin{tikzcd}
 & & M \times_B N \arrow[dl, "\widetilde{g}_1"'] \arrow[dr, "\widetilde{f}_2"] \arrow[ddll, bend right, "f_1 \circ \widetilde{g}_1"'] \arrow[ddrr, bend left, "g_2 \circ \widetilde{f}_2"] \arrow[dddd, dashed, red, bend right, "\exists ! \psi"'] & & \\
 & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\varphi_1"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] \arrow[dd, "\varphi_2"] & \\
A & & B & & C \\
 & M' \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] & & N' \arrow[ul, "g_1'"'] \arrow[ur, "g_2'"] & \\
 & & M' \times_B N' \arrow[ul, "\widetilde{g}_1'"] \arrow[ur, "\widetilde{f}_2'"'] \arrow[uull, bend left, "f_1' \circ \widetilde{g}_1'"] \arrow[uurr, bend right, "g_2' \circ \widetilde{f}_2'"'] & &
\end{tikzcd}$$

such that

$$\widetilde{g}_1' \circ \psi = \varphi_1 \circ \widetilde{g}_1 \qquad \text{and} \qquad \widetilde{f}_2' \circ \psi = \varphi_2 \circ \widetilde{f}_2.$$

From these equalities, we deduce that

$$f_1' \circ \widetilde{g}_1' \circ \psi = f_1' \circ \varphi_1 \circ \widetilde{g}_1 = f_1 \circ \widetilde{g}_1 \qquad \text{and} \qquad g_2' \circ \widetilde{f}_2' \circ \psi = g_2' \circ \varphi_2 \circ \widetilde{f}_2 = g_2 \circ \widetilde{f}_2.$$

Therefore, we define

$$\varphi_2 * \varphi_1 := \psi$$

as the horizontal composition of φ_1 and φ_2.

- Now that we have defined the horizontal compositions, let us check that they are functorial. Given a diagram of the form

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_g"] & B \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_f"] & C,
\end{tikzcd}
$$

with $f = (f_1, f_2)$ and $g = (g_1, g_2)$, we have to check that

$$\mathrm{id}_f * \mathrm{id}_g = \mathrm{id}_{f \circ g}\,.$$

We recall that $\mathrm{id}_f * \mathrm{id}_g$ is the unique morphism such that the red diagram

$$
\begin{tikzcd}
 & & M \times_B N \arrow[dl, red, "\widetilde{g}_1"'] \arrow[dr, red, "\widetilde{f}_2"] \arrow[ddll, bend right=40, "f_1 \circ \widetilde{g}_1"'] \arrow[ddrr, bend left=40, "g_2 \circ \widetilde{f}_2"] \arrow[dddd, red, dashed, "\exists!\ \mathrm{id}_f * \mathrm{id}_g"'] & & \\
 & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, red, "\mathrm{id}_M"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] \arrow[dd, red, "\mathrm{id}_N"] & \\
A & & B & & C. \\
 & M \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & & N \arrow[ul, "g_1"'] \arrow[ur, "g_2"'] & \\
 & & M \times_B N \arrow[ul, red, "\widetilde{g}_1"] \arrow[ur, red, "\widetilde{f}_2"'] \arrow[uull, bend left=40, "f_1 \circ \widetilde{g}_1"] \arrow[uurr, bend right=40, "g_2 \circ \widetilde{f}_2"'] & &
\end{tikzcd}
$$

is commutative, that is, such that

$$\widetilde{g}_1 \circ (\mathrm{id}_f * \mathrm{id}_g) = \widetilde{g}_1 \qquad \text{and} \qquad \widetilde{f}_2 \circ (\mathrm{id}_f * \mathrm{id}_g) = \widetilde{f}_2.$$

Now, by construction, $\mathrm{id}_{f \circ g}$ is just $\mathrm{id}_{M \times_B N}$, and this morphism clearly verifies that

$$\widetilde{g}_1 \circ \mathrm{id}_{M \times_B N} = \widetilde{g}_1 \qquad \text{and} \qquad \widetilde{f}_2 \circ \mathrm{id}_{M \times_B N} = \widetilde{f}_2.$$

Thus, by uniqueness,

$$\mathrm{id}_f * \mathrm{id}_g = \mathrm{id}_{M \times_B N} = \mathrm{id}_{f \circ g}\,.$$

Finally, given a diagram of the form

$$
\begin{tikzcd}[row sep=large]
A \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \beta"] \arrow[r, phantom, shift right=3ex, "\Downarrow \beta'"] & B \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow \alpha'"] & C,
\end{tikzcd}
$$

we need to check that the two ways to compose such diagram are equal, that is, we have to verify that

$$(\alpha' \circ \alpha) * (\beta' \circ \beta) = (\alpha' * \beta') \circ (\alpha * \beta).$$

As before, this is just a matter of checking diagrams, so we leave it as an exercise.

- The associator, the left unitor and the right unitor are defined by means of the universal property of pullbacks, and the rest of the bicategory axioms also follow mainly from the universal property of pullbacks.

When considering spans of discrete groupoids, one of the first issues that one may encounter is the computation of pullbacks. This is why, before focusing on our particular kind of spans, we devote a few pages to characterize how do spans in a slice category look like.

We start by showing a straightforward but useful fact: a span is a pullback of a given cospan if and only if it is isomorphic, in the bicategory of spans, to some pullback model of such a cospan.

Lemma 3.2.2. *Let $\mathcal{C}$ be a category with terminal objects and let $T' \in \mathcal{C}$ be a terminal object. Let $T \in \mathcal{C}$ be an object. Then,*

$$T \text{ is a terminal object in } \mathcal{C} \iff T \cong T'.$$

Proof. The direct implication is obvious: terminal objects are unique up to isomorphism. Nevertheless, let us recall the proof. Since both T and T' are terminal objects, we can deduce the existence of morphisms

$$\varphi_1 \colon T \longrightarrow T', \quad \varphi_2 \colon T' \longrightarrow T.$$

Moreover, since $\varphi_1 \circ \varphi_2$ and $\mathrm{id}_{T'}$ both belong to $\mathcal{C}(T', T')$ and T' is terminal, we must have that

$$\varphi_1 \circ \varphi_2 = \mathrm{id}_{T'} .$$

Analogously,

$$\varphi_2 \circ \varphi_1 = \mathrm{id}_{T} .$$

Thus, T and T' are isomorphic in $\mathcal{C}$. Conversely, assume that $T \cong T'$ and let

$$\varphi \colon T \longrightarrow T'$$

be an isomorphism. Let $A \in \mathcal{C}$ be any object. Since T' is terminal, there exists a unique morphism

$$f \colon A \longrightarrow T',$$

and composing with φ^{-1} we obtain a morphism

$$\varphi^{-1} \circ f \colon A \longrightarrow T.$$

Let us see that this is the unique morphism between A and T: if $A \xrightarrow{g} T$ is another morphism, we get that $\varphi \circ g$ is a morphism from A to T'. But T' is terminal, so we must have that $\varphi \circ g = f$ which, since φ is an isomorphism, is the same as $g = \varphi^{-1} \circ f$. ∎

Lemma 3.2.3. *Let $\mathcal{C}$ be a category with pullbacks and let*

$$A \xrightarrow{\ f\ } B \xleftarrow{\ g\ } C \tag{3.1}$$

be a cospan in $\mathcal{C}$. Then, its pullback is a terminal object in some category.

Proof. We construct a new category $\mathbf{Sq}(f,g)$ defined as follows: its objects are spans in $\mathcal{C}$

$$A \xleftarrow{g'} P \xrightarrow{f'} C$$

such that the square

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is commutative, and a morphism

$$\left(A \xleftarrow{g''} \widetilde{P} \xrightarrow{f''} C \right) \xrightarrow{\varphi} \left(A \xleftarrow{g'} P \xrightarrow{f'} C \right)$$

is a morphism $\varphi\colon \widetilde{P} \longrightarrow P$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} \widetilde{P} \arrow[dr, "\varphi"] \arrow[drr, bend left, "f''"] \arrow[ddr, bend right, "g''"'] & & \\ & P \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

is commutative. Thus, by definition, a pullback of 3.1 in $\mathcal{C}$ is just a terminal object in $\mathbf{Sq}(f,g)$. ■

Corollary 3.2.4. *Let $\mathcal{C}$ be a category with pullbacks and let*

$$A \xrightarrow{f} B \xleftarrow{g} C \tag{3.2}$$

be a cospan in $\mathcal{C}$. Then, a span in $\mathcal{C}$ is a pullback of 3.2 if and only if it is isomorphic, in the bicategory $\mathbf{Span}(\mathcal{C})$, to some pullback model of 3.2.

We now give the desired characterization of pullbacks in a slice category. As one can expect, the resulting pullback is essentially the pullback in the original category.

Proposition 3.2.5. *Let $\mathcal{C}$ be a category with pullbacks and let $X \in \mathcal{C}$ be an object in $\mathcal{C}$. Consider the cospan*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "D_A"'] & B \arrow[d, "D_B"] & C \arrow[l, "g"'] \arrow[d, "D_C"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd} \tag{3.3}$$

in the slice category $\mathcal{C}/X$*. Then, the span in* $\mathcal{C}/X$ *given by*

$$\begin{tikzcd} A \arrow[d, "D_A"'] & A \times_B C \arrow[l, "\widetilde{g}"'] \arrow[r, "\widetilde{f}"] \arrow[d, "D_{A\times_B C}"] & C \arrow[d, "D_C"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd} \tag{3.4}$$

is a pullback of 3.3, where

$$A \xleftarrow{\widetilde{g}} A \times_B C \xrightarrow{\widetilde{f}} C \tag{3.5}$$

is a pullback in $\mathcal{C}$ *of the cospan*

$$A \xrightarrow{f} B \xleftarrow{g} C,$$

and the morphism

$$d_{A\times_B C} \colon A \times_B C \longrightarrow X$$

is uniquely determined by the span 3.4. Equivalently, the span 3.4 is isomorphic, within the bicategory $\mathbf{Span}(\mathcal{C}/X)$*, to some pullback span of 3.3.*

Proof. A pullback in the slice $\mathcal{C}/X$ of the cospan 3.3 is a span

$$\begin{tikzcd} A \arrow[d, "D_A"'] & P \arrow[l, "\widetilde{g}"'] \arrow[r, "\widetilde{f}"] \arrow[d, "D_P"] & C \arrow[d, "D_C"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X, \end{tikzcd}$$

in $\mathcal{C}/X$ such that the cube

$$\begin{tikzcd} & X \arrow[rr, equal] \arrow[dd, equal] & & X \arrow[dd, equal] \\ P \arrow[ur, "D_P"] \arrow[rr, "\widetilde{f}"] \arrow[dd, "\widetilde{g}"'] & & C \arrow[ur, "D_C"] \arrow[dd, "g"] & \\ & X \arrow[rr, equal] & & X \\ A \arrow[ur, "D_A"] \arrow[rr, "f"] & & B \arrow[ur, "D_B"] & \end{tikzcd} \tag{3.6}$$

is commutative and such that for any other span in $\mathcal{C}/X$

$$\begin{tikzcd} A \arrow[d, "D_A"'] & P' \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_{P'}"] & C \arrow[d, "D_C"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd}$$

such that the blue cube

$$
\begin{tikzcd}
 & X \arrow[ddrrrr, equal, bend left] \arrow[ddddrr, equal] & & & & \\
P' \arrow[ur, "D_{P'}"] \arrow[ddrrrr, bend left, "q"] \arrow[ddddrr, bend right, "p"'] & & & & & \\
 & & & X \arrow[rr, equal] \arrow[dd, equal] & & X \arrow[dd, equal] \\
 & & P \arrow[ur, "D_P"] \arrow[rr, "\widetilde{f}"] \arrow[dd, "\widetilde{g}"'] & & C \arrow[ur, "D_C"] \arrow[dd, "g"] & \\
 & & & X \arrow[rr, equal] & & X \\
 & & A \arrow[ur, "D_A"] \arrow[rr, "f"] & & B \arrow[ur, "D_B"] &
\end{tikzcd}
\tag{3.7}
$$

is commutative, there exists a unique morphism

$$
\begin{tikzcd}
P' \arrow[r, "\varphi"] \arrow[d, "D_{P'}"'] & P \arrow[d, "D_P"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}
$$

in $\mathcal{C}/X$ such that the red diagram

$$
\begin{tikzcd}
 & X \arrow[ddrrrr, equal, bend left] \arrow[ddddrr, equal] \arrow[ddrr, equal] & & & & \\
P' \arrow[ur, "D_{P'}"] \arrow[ddrrrr, bend left, "q"] \arrow[ddddrr, bend right, "p"'] \arrow[ddrr, dashed, "\exists!\,\varphi"] & & & & & \\
 & & & X \arrow[rr, equal] \arrow[dd, equal] & & X \arrow[dd, equal] \\
 & & P \arrow[ur, "D_P"] \arrow[rr, "\widetilde{f}"] \arrow[dd, "\widetilde{g}"'] & & C \arrow[ur, "D_C"] \arrow[dd, "g"] & \\
 & & & X \arrow[rr, equal] & & X \\
 & & A \arrow[ur, "D_A"] \arrow[rr, "f"] & & B \arrow[ur, "D_B"] &
\end{tikzcd}
\tag{3.8}
$$

commutes. We will prove our claim by showing that the span 3.4 is a model for the pullback of the cospan 3.3. First of all, we have to determine the dimension morphism $D_{A \times_B C}$. We see that it exists if and only if

$$D_A \circ \widetilde{g} = D_C \circ \widetilde{f}.$$

If that is the case, $D_{A\times_B C}$ is obviously unique. Since 3.3 is a cospan and 3.5 is a pullback, we have the following equalities

$$\begin{cases} D_B \circ f = D_A, \\ D_B \circ g = D_C, \\ g \circ \widetilde{f} = f \circ \widetilde{g}. \end{cases}$$

Hence,

$$D_A \circ \widetilde{g} = D_B \circ f \circ \widetilde{g} = D_B \circ g \circ \widetilde{f} = D_C \circ \widetilde{f}.$$

Now that we have our well-defined span in $\mathbf{Span}(\mathcal{C}/X)$, it is time to see that it verifies the universal property. It obviously makes the diagram 3.6 commutative. Consider now any other span such as in 3.7. Since 3.5 is a pullback in $\mathcal{C}$, there must exist a unique morphism $\varphi\colon P' \longrightarrow A\times_B C$ such that the diagram

$$\begin{tikzcd}
P' \arrow[dr, dashed, "\exists!\,\varphi"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& A\times_B C \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] & C \arrow[d, "g"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd} \tag{3.9}$$

commutes. What is left now is to show that φ is a morphism in the slice category, that is, we have to check that the following diagram commutes

$$\begin{tikzcd}
P' \arrow[r, "\varphi"] \arrow[d, "D_{P'}"'] & A\times_B C \arrow[d, "D_{A\times_B C}"] \\
X \arrow[r, equal] & X.
\end{tikzcd} \tag{3.10}$$

Looking at 3.7, we can extract the equation $D_A \circ p = D_{P'}$, and from 3.9 it follows that $\widetilde{g}\circ\varphi = p$. Combining these equations, we get that

$$D_A \circ p = D_{P'} \iff D_A \circ \widetilde{g} \circ \varphi = D_{P'} \iff D_{A\times_B C} \circ \varphi = D_{P'},$$

which is the same as saying that 3.10 commutes. Thus, the span 3.4 is a pullback of 3.3 in $\mathcal{C}/X$.

■

Observe that, in practice, we do not know how to perform the composition of spans, since such a composition is defined by choosing a concrete pullback span using the axiom of choice. But it turns out that, if we choose two different models for the same pullback when doing a composition of spans, the two resulting compositions are isomorphic as spans. This is just an immediate consequence of corollary 3.2.4.

Thus, this fact motivates the following definition: given a category $\mathcal{C}$, we define the category $\overline{\mathbf{Span}}(\mathcal{C})$ which consists of the same objects as $\mathbf{Span}(\mathcal{C})$ but replacing spans by equivalence classes of spans modulo isomorphism. The composition of isomorphism classes is defined in the natural way

$$\left[\begin{tikzcd} & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ B & & C \end{tikzcd}\right] \circ \left[\begin{tikzcd} & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ A & & B \end{tikzcd}\right] :=$$

$$\left[\begin{tikzcd} & & M \times_B N \arrow[dl, "\widetilde{g}_1"'] \arrow[dr, "\widetilde{f}_2"] \arrow[ddll, bend right, "f_1\circ\widetilde{g}_1"'] \arrow[ddrr, bend left, "g_2\circ\widetilde{f}_2"] & & \\ & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ A & & B & & C \end{tikzcd}\right],$$

where the chosen pullback is not relevant, as we discussed earlier. This operation is indeed well-defined.

Proposition 3.2.6. *Consider the following pairs of isomorphic spans in a category $\mathcal{C}$ with pullbacks*

$$\begin{tikzcd} & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\varphi_1"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] \arrow[dd, "\varphi_2"] & \\ A & & B & & C, \\ & M' \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] & & N' \arrow[ul, "g_1'"] \arrow[ur, "g_2'"'] & \end{tikzcd}$$

where φ_1 and φ_2 are isomorphisms in $\mathcal{C}$. Then, we have that the compositions

$$\begin{tikzcd} & & M \times_B N \arrow[dl, "\widetilde{g}_1"'] \arrow[dr, "\widetilde{f}_2"] & & \\ & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\varphi_1"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] \arrow[dd, "\varphi_2"] & \\ A & & B & & C. \\ & M' \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] & & N' \arrow[ul, "g_1'"] \arrow[ur, "g_2'"'] & \\ & & M' \times_B N' \arrow[ul, "\widetilde{g}_1'"] \arrow[ur, "\widetilde{f}_2'"'] & & \end{tikzcd}$$

are also isomorphic as spans.

Proof. Since horizontal composition is functorial, the composition $\varphi_2 * \varphi_1$

$$\begin{tikzcd}
 & & M \times_B N \arrow[dl, "\widetilde{g_1}"'] \arrow[dr, "\widetilde{f_2}"] \arrow[ddll, bend right, "f_1 \circ \widetilde{g_1}"'] \arrow[ddrr, bend left, "g_2 \circ \widetilde{f_2}"] \arrow[dddd, dashed, red, bend right, "\varphi_2 * \varphi_1"'] & & \\
 & M \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\varphi_1"] & & N \arrow[dl, "g_1"'] \arrow[dr, "g_2"] \arrow[dd, "\varphi_2"] & \\
A & & B & & C \\
 & M' \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] & & N' \arrow[ul, "g_1'"'] \arrow[ur, "g_2'"] & \\
 & & M' \times_B N' \arrow[ul, "\widetilde{g_1'}"] \arrow[ur, "\widetilde{f_2'}"'] \arrow[uull, bend left, "f_1' \circ \widetilde{g_1'}"] \arrow[uurr, bend right, "g_2' \circ \widetilde{f_2'}"'] & &
\end{tikzcd}$$

provides the desired isomorphism. ∎

In order to simplify the notation, given an isomorphism class of spans

$$\left[A \xleftarrow{\ f\ } B \xrightarrow{\ g\ } C \right],$$

we usually drop the brackets and write

$$A \xleftarrow{\ f\ } B \xrightarrow{\ g\ } C$$

whenever this does not cause any confusion.

Throughout our work, we will consider spans of discrete groupoids over $\mathbb{Z}$ modulo isomorphism. Since composition defined in this way is pullback-model independent, we will always work with the standard model, unless otherwise stated. Using proposition 3.2.5, we only need to give a definition for the standard model in **DiscGrpd**. That is, given a cospan in **DiscGrpd**

$$A \xleftarrow{\ f\ } B \xrightarrow{\ g\ } C,$$

the standard model for its pullback

$$\begin{tikzcd}
A \times_B C \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is given by the discrete groupoid $A \times_B C$, defined on objects as

$$A \times_B C := \{(a, c) \in A \times C : f(a) = g(c)\} \subseteq A \times C$$

and, for every $(a_1, c_1), (a_2, c_2) \in A \times_B C$,

$$A \times_B C((a_1, c_1), (a_2, c_2)) = \{(\alpha, \beta) \in A(a_1, a_2) \times C(c_1, c_2) : f(\alpha) = g(\beta)\}\,.$$

The functors $\widetilde{g}$ and $\widetilde{f}$ are the canonical projections.

3.2.2 Construction of the Functor

Now that we know how the span category works, we can proceed with our work. As we have said at the beginning of the section, we are now interested in extending the bijective map from proposition 3.1.5 to a functor from spans of discrete groupoids over $\mathbb{Z}$ to **Cell**. In order to construct such a functor, the first thing we have to do is to figure out how a span in **DiscGrpd**$/\mathbb{Z}$ induces a cell-map. Let us fix a span in **DiscGrpd**$/\mathbb{Z}$

$$\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd} \tag{3.11}$$

We want to produce a cell-map from $\Psi(C)$ to $\Psi(D)$, that is, a map

$$\Psi(p,q)\colon \Psi(C) \longrightarrow \overline{\mathbf{Set}}/\Psi(D).$$

Inspired by our familiarity with linear functors (see [GCKT18d]), it is really natural to proceed as follows: consider the induced span in **Set**

$$\Psi(C) \xleftarrow{p} \Psi(M) \xrightarrow{q} \Psi(D), \tag{3.12}$$

that is, the span induced by applying the forgetful functor **DiscGrpd**$/\mathbb{Z} \longrightarrow$ **Set** to the span 3.11. Given the fact that every $x \in \Psi(C)$ can be thought as an arrow

$$\begin{aligned} \overline{x}\colon \{*\} &\longrightarrow \Psi(C) \\ * &\longmapsto x, \end{aligned}$$

we can proceed as in linear functors and define $\Psi(p,q)(x)$ as the submultiset of $\Psi(D)$

$$\begin{tikzcd} \{*\} \arrow[d, "\overline{x}"'] & p^{-1}(x) \arrow[l, "\widetilde{p}"'] \arrow[d, "\widetilde{\overline{x}}"] \arrow[dl, phantom, "\lrcorner"] \arrow[dr, dashed, bend left, "\Psi(p,q)(x)"] & \\ \Psi(C) & \Psi(M) \arrow[l, "p"] \arrow[r, "q"'] & \Psi(D) \end{tikzcd}$$

obtained via pullback along p and postcomposition with q. Thus, our natural candidate is

$$\begin{aligned} \Psi(p,q)\colon \Psi(C) &\longrightarrow \overline{\mathbf{Set}}/\Psi(D) \\ x &\longmapsto \Psi(p,q)(x) := (q(m))_{m\in p^{-1}(x)}. \end{aligned}$$

First of all, we observe that $\Psi(p,q)(x)$ does not necessarily need to be a finite submultiset of $\Psi(D)$. Therefore, we are forced to add finiteness conditions on our spans.

Definition 3.2.7. We say that a functor $p\colon M \longrightarrow C$ between discrete groupoids is *finite* if, for every $x \in C$, the cardinality of the fiber $p^{-1}(x)$ is finite. Then, we say that a span

$$\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

in **DiscGrpd**$/\mathbb{Z}$ is *finite* if the map p is finite.

Thus, finite spans induce well-defined maps. From now on, unless otherwise stated, it will be assumed that all the spans that we will consider are **finite**. Later on, we will show that adding this condition we obtain a subcategory of the category of spans.

Precisely because we are working with spans of discrete groupoids over $\mathbb{Z}$, we can ensure that $\Psi(p,q)$ preserves dimensions. Indeed, let $x \in \Psi(C)$ and let $y \in \Psi(p,q)(x)$, which means that $y = q(m)$ for some $m \in p^{-1}(x)$. Since 3.11 is commutative

$$\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}, \end{tikzcd}$$

we have that

$$D_C(x) = D_C(p(m)) = D_M(m) = D_D(q(m)) = D_D(y),$$

hence $\Psi(p,q)$ preserves dimensions without further conditions. The problem is that adding the finiteness condition is not enough, since we cannot guarantee that the equivalence relations are respected. At this point, the only way to proceed is to modify the category $\overline{\mathbf{Span}}(\mathbf{DiscGrpd}/\mathbb{Z})$ so as to ensure that $\Psi(p,q)$ is a cell-map.

Before attacking this problem, we make an important observation about how the composition of these induced cell-maps works. Given a pair of composable spans in **DiscGrpd**$/\mathbb{Z}$

$$\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[d, "D_D"'] & N \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d, "D_N"] & E \arrow[d, "D_E"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

and given any $x \in \Psi(C)$, one can naturally consider the following diagram

$$\begin{tikzcd} \{*\} \arrow[d, "\overline{x}"'] & p^{-1}(x) \arrow[l, "\widetilde{p}"'] \arrow[r, equal] \arrow[d, "\widetilde{\overline{x}}"] \arrow[dl, phantom, "\llcorner"] & p^{-1}(x) \arrow[d, "\Psi(p,q)(x)"] & p^{-1}(x) \times_{\Psi(D)} \Psi(N) \arrow[l, "\widetilde{p}'"'] \arrow[d, "\widetilde{\Psi(p,q)(x)}"] \arrow[dl, phantom, "\llcorner"] \arrow[dr, dashed, bend left, red, "q' \circ \widetilde{\Psi(p,q)(x)}"] & \\ \Psi(C) & \Psi(M) \arrow[l, "p"'] \arrow[r, "q"] & \Psi(D) & \Psi(N) \arrow[l, "p'"'] \arrow[r, "q'"] & \Psi(E) \end{tikzcd}$$

where, once we compute $\Psi(p,q)(x)$, we consider the postcomposition with q' of its pullback along p'. The natural question is if the submultiset that we obtain performing these operations coincides with the one we obtain via ordinary composition of cell-maps, and the answer is affirmative. Given $\Psi(p',q')$, we define its *linear extension* $\Psi^L(p',q')$ as

$$\begin{aligned} \Psi^L(p',q') \colon \overline{\mathbf{Set}}/\Psi(D) &\longrightarrow \overline{\mathbf{Set}}/\Psi(E) \\ I \xrightarrow{f} \Psi(D) &\longmapsto q' \circ \widetilde{f}, \end{aligned}$$

where $\widetilde{f}$ is the pullback along p' in **Set**, using the standard model. Thus, it makes sense to consider the ordinary composition of maps

$$\Psi^L(p',q') \circ \Psi(p,q) \colon \Psi(C) \longrightarrow \overline{\mathbf{Set}}/\Psi(D) \longrightarrow \overline{\mathbf{Set}}/\Psi(E).$$

Proposition 3.2.8. *The composition $\Psi(p',q')\circ\Psi(p,q)$ as cell-maps coincides with the ordinary composition $\Psi^L(p',q')\circ\Psi(p,q)$ of $\Psi(p,q)$ with the linear extension of $\Psi(p',q')$.*

Proof. Let $x\in\Psi(C)$. We have, on the one hand, that

$$(\Psi(p',q')\circ\Psi(p,q))(x)=\bigcup_{m\in p^{-1}(x)}\Psi(p',q')(q(m))=\bigcup_{m\in p^{-1}(x)}(q'(n))_{n\in p'^{-1}(q(m))}=$$

$$\coprod_{m\in p^{-1}(x)}p'^{-1}(q(m))\xrightarrow{q'\circ i}\Psi(E),$$

where

$$i\colon \coprod_{m\in p^{-1}(x)}p'^{-1}(q(m))\longrightarrow\Psi(N)$$

is the inclusion. On the other hand, we know that

$$(\Psi^L(p',q')\circ\Psi(p,q))(x)=\ p^{-1}(x)\times_{\Psi(D)}\Psi(N)\xrightarrow{q'\circ\Psi\widetilde{(p,q)(x)}}\Psi(E).$$

Since

$$p^{-1}(x)\times_{\Psi(D)}\Psi(N)=\left\{(m,n)\in\Psi(M)\times\Psi(N):m\in p^{-1}(x),\,p'(n)=q(m)\right\},$$

there is a natural bijection

$$h\colon\begin{array}{rcl}\coprod_{m\in p^{-1}(x)}p'^{-1}(q(m)) & \longrightarrow & p^{-1}(x)\times_{\Psi(D)}\Psi(N)\\ n & \longmapsto & (m,n),\text{ if } n\in p'^{-1}(q(m))\end{array}$$

that trivially makes the following triangle commutative

$$\begin{tikzcd}
\coprod_{m\in p^{-1}(x)}p'^{-1}(q(m)) \arrow[rr, "h"] \arrow[dr, "q'\circ i"'] & & p^{-1}(x)\times_{\Psi(D)}\Psi(N) \arrow[dl, "q'\circ\Psi\widetilde{(p,q)(x)}"] \\
& \Psi(E). &
\end{tikzcd}$$

Therefore,

$$(\Psi(p',q')\circ\Psi(p,q))(x)=(\Psi^L(p',q')\circ\Psi(p,q))(x),$$

as desired. ■

Let us now come back to the conditions that we have to add to $\overline{\mathbf{Span}}(\mathbf{DiscGrpd}/\mathbb{Z})$. Notice that we can formulate such conditions in a simpler way using the fact that

$$\left(\ \Psi(C)\xrightarrow{\Psi(p,q)}\overline{\mathbf{Set}}/\Psi(D)\right)=\left(\Psi(C)\xrightarrow{\Psi(p,\mathrm{id})}\Psi(M)\xrightarrow{\Psi(\mathrm{id},q)}\Psi(D)\right),$$

that is,

$$\Psi(p,q)=\Psi(\mathrm{id}_{\Psi(M)},q)\circ\Psi(p,\mathrm{id}_{\Psi(M)}).$$

Indeed, for every $x \in \Psi(C)$,

$$\Psi(\mathrm{id}_{\Psi(M)}, q) \circ \Psi(p, \mathrm{id}_{\Psi(M)})(x) = \bigcup_{m \in p^{-1}(x)} \{q(m)\} = (q(m))_{m \in p^{-1}(x)} = \Psi(p, q).$$

This fact may be visualized even more easily using the linear extension of $\Psi(\mathrm{id}_{\Psi(M)}, q)$ and proposition 3.2.8. Observe that we can rewrite the span 3.12 as

$$\left(\Psi(C) \xleftarrow{\;p\;} \Psi(M) \xrightarrow{\;q\;} \Psi(D)\right) =$$

$$\left(\Psi(C) \xleftarrow{\;p\;} \Psi(M) = \Psi(M) = \Psi(M) \xrightarrow{\;q\;} \Psi(D)\right),$$

and therefore given any $x \in \Psi(C)$,

$$\overbrace{\begin{tikzcd}
\{*\} \arrow[d, "\overline{x}"'] & p^{-1}(x) \arrow[l, "\widetilde{p}"'] \arrow[d, "\widetilde{\overline{x}}"] \arrow[dl, phantom, very near start, "\llcorner"] \arrow[dr, dashed, bend left, red, "\Psi(p,q)(x)"] & \\
\Psi(C) & \Psi(M) \arrow[l, "p"] \arrow[r, "q"'] & \Psi(D)
\end{tikzcd}}^{\Psi(p,q)} =$$

$$\overbrace{\begin{tikzcd}
\{*\} \arrow[d, "\overline{x}"'] & p^{-1}(x) \arrow[l, "\widetilde{p}"'] \arrow[d, "\widetilde{\overline{x}}"] \arrow[dl, phantom, very near start, "\llcorner"] \arrow[dr, dashed, bend left, "\widetilde{\overline{x}}"'] \\
\Psi(C) & \Psi(M) \arrow[l, "p"']
\end{tikzcd}}^{\Psi(p,\mathrm{id}_{\Psi(M)})}
\overbrace{\begin{tikzcd}
p^{-1}(x) \arrow[r, equal] \arrow[d, "\widetilde{\overline{x}}"] & p^{-1}(x) \arrow[d, "\widetilde{\overline{x}}"] \arrow[dl, phantom, very near start, "\llcorner"] \arrow[dr, dashed, bend left, red, "\Psi(p,q)(x)"] & \\
\Psi(M) \arrow[r, equal] & \Psi(M) \arrow[r, "q"] & \Psi(D).
\end{tikzcd}}^{\Psi^{L}(\mathrm{id}_{\Psi(M)},q)}$$

Hence, we have broken down the problem into two smaller subproblems. The map

$$\begin{aligned} \Psi(\mathrm{id}_{\Psi(M)}, q) \colon \Psi(M) &\longrightarrow \Psi(D) \\ x &\longmapsto \{q(x)\} \end{aligned}$$

is clearly a cell-map. Indeed, it preserves relations because of the following simple reason: take $x, x' \in \Psi(M)$ and assume that $x \sim x'$. Within the groupoid M, this means that there exists an arrow $x \longrightarrow x'$ and, by functoriality of q, there is an arrow $q(x) \longrightarrow q(x')$. This means exactly that $q(x) \sim q(x')$ in $\Psi(D)$, so

$$\Psi(\mathrm{id}_{\Psi(M)}, q)(x) = \{q(x)\} \sim \{q(x')\} = \Psi(\mathrm{id}_{\Psi(M)}, q)(x')$$

as submultisets of $\Psi(D)$. Thus it all reduces to impose that the map

$$\begin{aligned} \Psi(p, \mathrm{id}_{\Psi(M)}) \colon \Psi(C) &\longrightarrow \Psi(M) \\ x &\longmapsto p^{-1}(x) \end{aligned}$$

is a cell-map. The following definition encodes the necessary and sufficient condition.

Definition 3.2.9. We say that a functor $f\colon D \longrightarrow C$ between discrete groupoids has the *p-property* if, for every $x, y \in C$,

$$x \sim y \implies f^{-1}(x) \sim f^{-1}(y)$$

as submultisets of $\mathrm{Obj}(D)$. We also say that a span in $\mathbf{DiscGrpd}/\mathbb{Z}$

$$\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

has the *p-property* if the map $p\colon M \longrightarrow C$ has the p-property.

The p-property solves the problem we have, but we cannot add it to $\overline{\mathbf{Span}}(\mathbf{DiscGrpd}/\mathbb{Z})$ without checking that the resulting data is also a category.

Proposition 3.2.10. *Let* $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$ *be the category* $\overline{\mathbf{Span}}(\mathbf{DiscGrpd}/\mathbb{Z})$ *together with the p-property and the finiteness condition, that is, for every span in* $\mathbf{DiscGrpd}/\mathbb{Z}$

$$\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}, \end{tikzcd}$$

all the spans within its isomorphism class

$$\left[\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right]$$

are assumed to verify the p-property and to be finite. Then, $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$ *is a subcategory of* $\overline{\mathbf{Span}}(\mathbf{DiscGrpd}/\mathbb{Z})$.

Proof. Let us first make the following important observation. By construction, it is clear that if we have a pair of isomorphic spans, one has the p-property if and only if the other has it, and the same holds for finite spans. Thus, whenever a span verifies the p-property and the finiteness condition, so do all members of its isomorphism class.

That said, the identity functor of a given discrete groupoid trivially satisfies the p-property and clearly all its fibers are finite, hence the identity spans are in

$$\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}).$$

It only remains to show that the composition of spans with the p-property and the finiteness condition gives also a span with the same characteristics. In order to do so, it suffices to check that the p-property is stable under pullbacks and that the composition of maps with the p-property also has the p-property, and the same for the finiteness condition. Thus, we want to see that if the cube

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
P \arrow[ur, "D_P"] \arrow[rr, "\widetilde{f}"] \arrow[dd, "\widetilde{g}"'] \arrow[drr, phantom, "\lrcorner", very near start] & & C \arrow[ur, "D_C"] \arrow[dd, "g"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
A \arrow[ur, "D_A"] \arrow[rr, "f"] & & B \arrow[ur, "D_B"] &
\end{tikzcd}$$

is a pullback in $\mathbf{DiscGrpd}/\mathbb{Z}$ with g having the p-property and being finite, then $\widetilde{g}$ has the p-property and it is finite as well. Using proposition 3.2.5, we can replace the cube by the pullback square

$$\begin{tikzcd}
P \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

in **DiscGrpd**. Let $a \sim a^{'}$ in A. Then, by functoriality of f, $f(a) \sim f(a')$. Since g verifies the p-property, we have that $g^{-1}(f(a)) \sim g^{-1}(f(a'))$. Now observe that, by definition (recall that we can always assume that we are working with the standard model),

$$\widetilde{g}^{-1}(a) = \{(a,c) : c \in C,\ f(a) = g(c)\}$$

and

$$\widetilde{g}^{-1}(a') = \{(a',c) : c \in C,\ f(a') = g(c)\}\,.$$

Therefore, we clearly have that

$$\begin{array}{lcl}
g^{-1}(f(a)) & = \{c \in C : g(c) = f(a)\} \sim & \widetilde{g}^{-1}(a) \\
\quad \wr & & \\
g^{-1}(f(a')) & = \{c \in C : g(c) = f(a')\} \sim & \widetilde{g}^{-1}(a'),
\end{array}$$

so by transitivity we get that $\widetilde{g}^{-1}(a) \sim \widetilde{g}^{-1}(a')$, as desired. This argument shows that $\widetilde{g}$ has the p-property, and a similar argument shows that it is also finite: for every $a \in A$, we have already seen that the fiber $\widetilde{g}^{-1}(a)$ is in one-to-one correspondence with the fiber $g^{-1}(f(a))$, which is finite by hypothesis. Hence, $\widetilde{g}$ is finite as well.

Finally, it is obvious that the composition of finite maps is finite, so we just need to check that the composition of maps having the p-property also has the p-property. Consider the maps

$$C \xrightarrow{g} D \xrightarrow{f} E,$$

with f and g verifying the p-property. Let $e, e' \in E$ such that $e \sim e'$. By hypothesis, we have that

$$(y_i)_{i\in I} = f^{-1}(e) \sim f^{-1}(e') = (y'_j)_{j\in J}.$$

This means that there exists a bijection $\varphi\colon I \longrightarrow J$ such that, for every $i \in I$,

$$y_i \sim y'_{\varphi(i)}.$$

Write

$$\begin{cases} g^{-1}(y_i) = (z^i_k)_{k\in K_i} \\ g^{-1}(y'_{\varphi(i)}) = (w^i_l)_{l\in L_i}. \end{cases}$$

Since g also has the p-property we have that, for every $i \in I$, there exists a bijection $\psi_i\colon K_i \longrightarrow L_i$ such that, for every $k \in K_i$,

$$z^i_k \sim w^i_{\psi_i(k)}.$$

Since the fibers of any map are disjoint and using the fact that the disjoint union of bijections is a bijection, it clearly follows that

$$(f\circ g)^{-1}(e) = g^{-1}(f^{-1}(e)) \sim g^{-1}(f^{-1}(e')) = (f\circ g)^{-1}(e')$$

as submultisets of $\mathrm{Obj}(C)$. ■

Summing up, our candidate to extend the map in 3.1.5 is the assignment

$$\begin{array}{rcl} \Psi\colon \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) & \longrightarrow & \mathbf{Cell} \\ C \xrightarrow{D_C} \mathbb{Z} & \longmapsto & \Psi(C) \\ \left[\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right] & \longmapsto & \Psi(p,q). \end{array}$$

Proposition 3.2.11. *The above-mentioned assignment is a well-defined functor.*

Proof. First of all, let us check that Ψ is well-defined. We have already seen that Ψ maps discrete groupoids and spans to cell-sets and cell-maps in **Cell**, respectively; so we only need to check that it is well-defined on isomorphism classes. Consider a pair of isomorphic spans in $\mathbf{DiscGrpd}/\mathbb{Z}$

$$\begin{tikzcd} & M \arrow[dl, "p"'] \arrow[d, "D_M"] \arrow[dr, "q"] \arrow[dd, bend left, "\varphi", "\cong"'] & \\ C \arrow[d, "D_C"'] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[dr, equal] & N \arrow[ul, "p'"] \arrow[ur, "q'"'] \arrow[d, "D_N"] & \mathbb{Z}. \arrow[dl, equal] \\ & \mathbb{Z} & \end{tikzcd}$$

This means that φ is an isomorphism of discrete groupoids over $\mathbb{Z}$ such that

$$\begin{cases} p' \circ \varphi = p, \\ q' \circ \varphi = q. \end{cases} \tag{3.13}$$

The corresponding cell-maps are

$$\begin{aligned} \Psi(p,q)\colon \Psi(C) &\longrightarrow \overline{\mathbf{Set}}/\Psi(D) \\ x &\longmapsto (q(m))_{m \in p^{-1}(x)} \end{aligned}$$

and

$$\begin{aligned} \Psi(p',q')\colon \Psi(C) &\longrightarrow \overline{\mathbf{Set}}/\Psi(D) \\ x &\longmapsto (q'(n))_{n \in p'^{-1}(x)}. \end{aligned}$$

Now, for every $x \in \Psi(C)$, we have the following commutative diagram

$$\begin{tikzcd}
\{*\} \arrow[d, "\overline{x}"'] & p^{-1}(x) \arrow[l, "\widetilde{p}"'] \arrow[d, "i"] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, dashed, red, bend left, "\Psi(p,q)(x)"] & \\
\Psi(C) \arrow[d, equal] & \Psi(M) \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\varphi"] & \Psi(D) \arrow[d, equal] \\
\Psi(C) & \Psi(N) \arrow[l, "p'"'] \arrow[r, "q'"] & \Psi(D) \\
\{*\} \arrow[u, "\overline{x}"] & p'^{-1}(x) \arrow[l, "\widetilde{p'}"'] \arrow[u, "j"'] \arrow[ul, phantom, "\ulcorner", very near start] \arrow[ur, dashed, red, bend right, "\Psi(p',q')(x)"'] &
\end{tikzcd}$$

where $\varphi\colon \Psi(M) \longrightarrow \Psi(N)$ is just the functor $\varphi\colon M \longrightarrow N$ restricted to objects, and therefore it yields a bijection between the sets $\Psi(M)$ and $\Psi(M)$. What we want to know now is if we can restrict this bijection to a bijection $\widetilde{\varphi}$ between the fibers so that the following diagram

$$\begin{tikzcd}
\Psi(M) \arrow[r, "\varphi"] & \Psi(M) \\
p^{-1}(x) \arrow[u, "i"] \arrow[r, "\widetilde{\varphi}"] & p'^{-1}(x) \arrow[u, "j"']
\end{tikzcd} \tag{3.14}$$

commutes. Of course, we just have to check that $\varphi(p^{-1}(x)) = p'^{-1}(x)$. From 3.13, we deduce that

$$p' \circ \varphi = p \iff p \circ \varphi^{-1} = p',$$

which implies that

$$\varphi(p^{-1}(x)) = (p \circ \varphi^{-1})^{-1}(x) = p'^{-1}(x),$$

as wanted. Hence, $\widetilde{\varphi}$ is a bijection between the fibers and, moreover, is such that the following triangle

$$\begin{tikzcd} p^{-1}(x) \arrow[rr, "\widetilde{\varphi}"] \arrow[dr, "\Psi(p,q)(x)"'] & & p'^{-1}(x) \arrow[dl, "\Psi(p',q')(x)"] \\ & \Psi(D) & \end{tikzcd}$$

is commutative. Indeed,

$$\Psi(p',q')(x) \circ \widetilde{\varphi} = q' \circ j \circ \widetilde{\varphi} = q' \circ \varphi \circ i = q \circ i = \Psi(p,q)(x),$$

where we have used the commutativity of 3.14 and the equations 3.13, respectively. Now we continue the proof with the functoriality. Let

$$\begin{tikzcd} C \arrow[d, "D_C"] \\ \mathbb{Z} \end{tikzcd} \in \mathbf{DiscGrpd}/\mathbb{Z}.$$

Then,

$$\Psi(\mathrm{id}_C) = \Psi\left(\begin{tikzcd} C \arrow[r, equal] \arrow[d, "D_C"'] & C \arrow[r, equal] \arrow[d, "D_C"] & C \arrow[d, "D_C"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) = \mathrm{id}_{\Psi(C)},$$

just by construction. Now comes the hard part: for every pair of composable spans

$$\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd} \qquad \begin{tikzcd} D \arrow[d, "D_D"'] & N \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d, "D_N"] & E \arrow[d, "D_E"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

we have to show that

$$\Psi\left(\left(\begin{tikzcd} D \arrow[d, "D_D"'] & N \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d, "D_N"] & E \arrow[d, "D_E"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \circ \left(\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right)\right) =$$

$$\Psi\left(\begin{tikzcd} D \arrow[d, "D_D"'] & N \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d, "D_N"] & E \arrow[d, "D_E"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \circ \Psi\left(\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right).$$

On the one hand, we have that

$$\left(\begin{tikzcd} D \arrow[d, "D_D"'] & N \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d, "D_N"] & E \arrow[d, "D_E"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \circ \left(\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) =$$

$$\begin{tikzcd}
 & & M \times_D N \arrow[dl, "\widetilde{p'}"'] \arrow[d, "D_{M\times_D N}"] \arrow[dr, "\widetilde{q}"] & & \\
C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[d, "D_M"'] \arrow[dr, "q"] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & N \arrow[r, "q'"] \arrow[d, "D_N"] \arrow[dl, "p'"'] & E \arrow[d, "D_E"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[dr, equal] & D \arrow[d, "D_D"] & \mathbb{Z} \arrow[r, equal] \arrow[dl, equal] & \mathbb{Z} \\
 & & \mathbb{Z} & &
\end{tikzcd} =$$

$$\begin{tikzcd}
C \arrow[d, "D_D"'] & M \times_D N \arrow[l, "p\circ\widetilde{p'}"'] \arrow[r, "q'\circ\widetilde{q}"] \arrow[d, "D_{M\times_D N}"] & E \arrow[d, "D_E"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}.
\end{tikzcd}$$

Therefore,

$$\Psi\left(\left(\begin{tikzcd}
D \arrow[d, "D_D"'] & N \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d, "D_N"] & E \arrow[d, "D_E"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}\right) \circ \left(\begin{tikzcd}
C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}\right)\right) =$$

$$\Psi\left(\begin{tikzcd}
C \arrow[d, "D_D"'] & M \times_D N \arrow[l, "p\circ\widetilde{p'}"'] \arrow[r, "q'\circ\widetilde{q}"] \arrow[d, "D_{M\times_D N}"] & E \arrow[d, "D_E"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}\right) =$$

$$\begin{aligned}
\Psi(p\circ\widetilde{p'}, q'\circ\widetilde{q}) \colon \Psi(C) &\longrightarrow \overline{\mathbf{Set}}/\Psi(E) \\
x &\longmapsto (q'(\widetilde{q}(m,n)))_{(m,n)\in(p\circ\widetilde{p'})^{-1}(x)} = (q'(n))_{(m,n)\in(p\circ\widetilde{p'})^{-1}(x)}.
\end{aligned}$$

The diagram representing $\Psi(p\circ\widetilde{p'}, q'\circ\widetilde{q})$ is, for every $x \in \Psi(C)$,

$$\begin{tikzcd}
\{*\} \arrow[d, "\overline{x}"'] & (p\circ\widetilde{p'})^{-1}(x) \arrow[l, "\widetilde{p\circ\widetilde{p'}}"'] \arrow[d, "\widetilde{\overline{x}}"] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, dashed, red, "\Psi(p\circ\widetilde{p'}, q'\circ\widetilde{q})(x) = (q'\circ\widetilde{q})\circ\widetilde{\overline{x}}"] & \\
\Psi(C) & \Psi(M \times_D N) \arrow[l, "p\circ\widetilde{p'}"] \arrow[r, "q'\circ\widetilde{q}"'] & \Psi(E).
\end{tikzcd}$$

On the other hand,

$$\Psi\left(\begin{tikzcd}
D \arrow[d, "D_D"'] & N \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[d, "D_N"] & E \arrow[d, "D_E"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}\right) \circ \Psi\left(\begin{tikzcd}
C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}\right) = \Psi(p', q') \circ \Psi(p, q),$$

where

$$\begin{aligned}
\Psi(p, q) \colon \Psi(C) &\longrightarrow \overline{\mathbf{Set}}/\Psi(D) \\
x &\longmapsto (q(m))_{m\in p^{-1}(x)}
\end{aligned}$$

and

$$\begin{array}{rcl} \Psi(p',q')\colon \Psi(D) & \longrightarrow & \overline{\mathbf{Set}}/\Psi(E) \\ y & \longmapsto & (q'(n))_{n\in p'^{-1}(y)}. \end{array}$$

Using proposition 3.2.8 we have that, for every $x \in \Psi(C)$,

$$(\Psi(p',q') \circ \Psi(p,q))(x) = (\Psi^L(p',q') \circ \Psi(p,q))(x) = q' \circ \Psi\widetilde{(p,q)}(x)$$

as indicated in the following diagram

$$\begin{tikzcd}
\{*\} \arrow[d, "\overline{x}"'] \arrow[dr, phantom, "\llcorner"] & p^{-1}(x) \arrow[l, "\widetilde{p}"'] \arrow[r, equal] \arrow[d, "\widetilde{\overline{x}}"] & p^{-1}(x) \arrow[d, "\Psi(p,q)(x)"] \arrow[dr, phantom, "\llcorner"] & p^{-1}(x) \times_{\Psi(D)} \Psi(N) \arrow[l, "\widetilde{p'}"'] \arrow[d, "\Psi\widetilde{(p,q)}(x)"] \arrow[dr, dashed, red, bend left, "q'\circ\Psi\widetilde{(p,q)}(x)"] & \\
\Psi(C) & \Psi(M) \arrow[l, "p"'] \arrow[r, "q"] & \Psi(D) & \Psi(N) \arrow[l, "p'"'] \arrow[r, "q'"] & \Psi(E).
\end{tikzcd}$$

Observe that

$$(p \circ \widetilde{p'})^{-1}(x) = \Big\{(m,n) \in \Psi(M \times_D N) : p(\widetilde{p'}(m,n)) = x\Big\} =$$

$$\{(m,n) \in \Psi(M) \times \Psi(N) : q(m) = p'(n),\, p(m) = x\} = p^{-1}(x) \times_{\Psi(D)} \Psi(N),$$

and

$$\Psi(p\circ\widetilde{p'}, q'\circ\widetilde{q})(x)(m,n) = (q'\circ\widetilde{q})(\widetilde{\overline{x}}(m,n)) = (q'\circ\widetilde{q})(m,n) = q'(n) = (q'\circ\Psi\widetilde{(p,q)}(x))(m,n).$$

Thus, equality holds and therefore Ψ is a functor. ■

Moreover, as we promised at the beginning of this chapter, it turns out that Ψ is a full and bijective on objects functor between $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbf{Z})$ and $\mathbf{Cell}$, but unfortunately it fails to be faithful, as we will show.

Proposition 3.2.12. *The functor*

$$\Psi\colon \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \longrightarrow \mathbf{Cell}$$

is full and bijective on objects.

Proof. We already know that Ψ is a bijection on objects, so it only remains to check that Ψ is full. Let

$$\begin{tikzcd} C \arrow[d, "D_C"'] \\ \mathbb{Z} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[d, "D_D"] \\ \mathbb{Z} \end{tikzcd}$$

be a pair of discrete groupoids over $\mathbb{Z}$. We want to verify that the induced map on arrows

$$\Psi = \Psi_{C,D}\colon \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})(C,D) \longrightarrow \mathbf{Cell}(\Psi(C),\Psi(D))$$

is surjective. Given any cell-map $f \in \mathbf{Cell}(\Psi(C), \Psi(D))$ we write, for every $x \in \Psi(C)$,

$$f(x) = (u_i^x)_{i \in I_x} = u^x \colon I_x \longrightarrow \Psi(D).$$

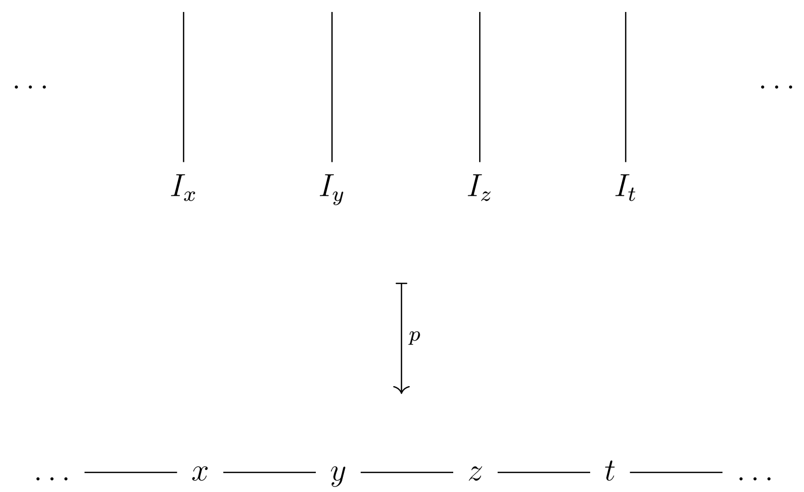
Consider the groupoid M defined on objects as

$$\mathrm{Obj}(M) := \coprod_{x \in C} I_x.$$

We want to define a functor $p \colon M \longrightarrow C$ between discrete groupoids. In objects, we define it as

$$\begin{array}{rcl} p \colon \coprod_{x \in C} I_x & \longrightarrow & \mathrm{Obj}(C) \\ i & \longrightarrow & p(i) = x, \text{ if } i \in I_x, \end{array}$$

that is, p is the projection



Of course, the fibers $p^{-1}(x) = I_x$ are finite because f is a cell-map, therefore the map p is finite. Now in order to define $\mathrm{Hom}(M)$, one has to take into account that we want p to verify the p-property, that is, for every $x, y \in C$ such that $x \sim y$, we must have that

$$p^{-1}(x) = I_x \sim I_y = p^{-1}(y).$$

Thus, given $x, y \in C$ with $x \sim y$, we know that $f(x) \sim f(y)$, so in particular there exists a bijection

$$\varphi_{x,y} \colon I_x \longrightarrow I_y.$$

Hence, for every $z \in I_x$, we add one and only one arrow between z and $\varphi_{x,y}(z)$ in such a way that M is a discrete groupoid. In other words, the hom-set of M is defined as the minimal hom-set such that p has the p-property and M is a discrete groupoid. For technical reasons, let us make the following choices: if $x, y \in C$ are such that $x \sim y$, we choose a bijection $\varphi_{x,y} \colon I_x \longrightarrow I_y$ in such a way that

$$\varphi_{y,x} = \varphi_{x,y}^{-1},$$

and if $x, y, z \in C$ are such that $x \sim y \sim z$, we choose $\varphi_{x,z}$ such that

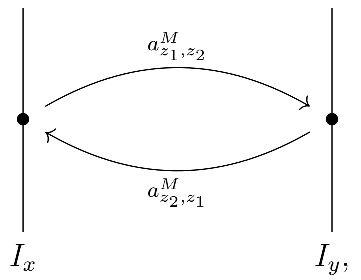
$$\varphi_{x,z} = \varphi_{y,z} \circ \varphi_{x,y}.$$

We proceed now to define $\mathrm{Hom}(M)$: given z_1 and $z_2 \in M$,

1. If $z_1, z_2 \in I_x$ for some $x \in C$, then
$$|\operatorname{Hom}_M(z_1, z_2)| = \begin{cases} 1, & \text{if } z_1 = z_2, \\ 0, & \text{otherwise.} \end{cases}$$

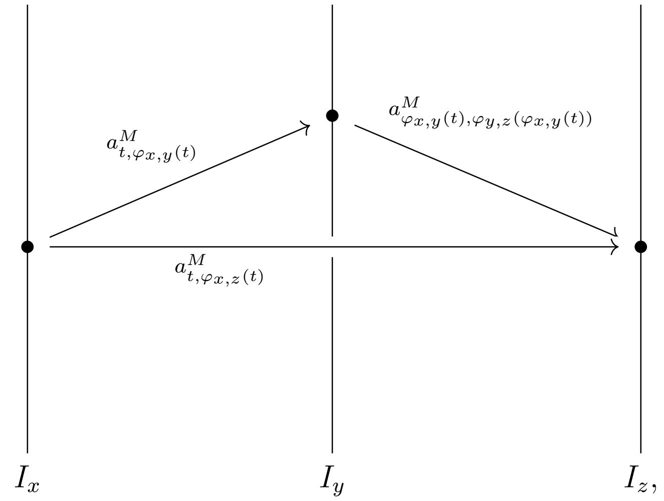
2. If $z_1 \in I_x$ and $z_2 \in I_y$ for some pair $x, y \in C$, we distinguish two cases:
   (a) If $x \not\sim y$, then
   $$|\operatorname{Hom}_M(z_1, z_2)| = 0.$$
   (b) If $x \sim y$, with $x \neq y$, then
   $$|\operatorname{Hom}_M(z_1, z_2)| = \begin{cases} 1, & \text{if } \varphi_{x,y}(z_1) = z_2, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that, equivalently, condition 1 can be included in $2(b)$ if we demand that $\varphi_{x,x} = \mathrm{id}_{I_x}$, for every $x \in C$. Thus, within each fiber I_x we just add the identities, and between two different fibers I_x and I_y we add no arrows between their elements if $x \not\sim y$, and if $x \sim y$ we add only one arrow between the corresponding elements under the chosen bijection $\varphi_{x,y} \colon I_x \longrightarrow I_y$. Observe that this construction is enough to define a discrete groupoid. Let $a^M_{z_1,z_2}$ be the unique arrow, if exists, between $z_1, z_2 \in M$. Since for every $x \sim y$ in C we have that $\varphi^{-1}_{x,y} = \varphi_{y,x}$, every arrow $a^M_{z_1,z_2}$ with $z_1 \in I_x$ and $z_2 = \varphi_{x,y}(z_1) \in I_y$ has associated a unique arrow $a^M_{z_2,z_1}$



and therefore we define
$$(a^M_{z_1,z_2})^{-1} := a^M_{z_2,z_1}.$$
We also have a well-defined composition: indeed, if $x \sim y \sim z$ in C, we have that the chosen bijections
$$\begin{tikzcd} I_x \arrow[r, "\varphi_{x,y}"] \arrow[rr, bend right, "\varphi_{x,z}"'] & I_y \arrow[r, "\varphi_{y,z}"] & I_z \end{tikzcd}$$
are such that
$$\varphi_{y,z} \circ \varphi_{x,y} = \varphi_{x,z}.$$

Therefore, for each $t \in I_x$ we always have a triangle



which allows us to define

$$a^M_{\varphi_{x,y}(t),\varphi_{y,z}(\varphi_{x,y}(t))} \circ a^M_{t,\varphi_{x,y}(t)} := a^M_{t,\varphi_{x,z}(t)}.$$

In this way, M is a discrete groupoid, and we can define $p\colon M \longrightarrow C$ on arrows as follows: for every $z_1, z_2 \in M$ such that $\mathrm{Hom}_M(z_1, z_2) \neq \varnothing$ (and therefore $\mathrm{Hom}_C(p(z_1), p(z_2)) \neq \varnothing$ by construction),

$$\begin{array}{rcl} p = p_{z_1,z_2}\colon \mathrm{Hom}_M(z_1, z_2) & \longrightarrow & \mathrm{Hom}_C(p(z_1), p(z_2)) \\ a^M_{z_1,z_2} & \longmapsto & a^C_{p(z_1),p(z_2)}, \end{array}$$

which gives indeed a functor using proposition 3.1.2. Finally, we define a functor $q\colon M \longrightarrow D$ in the following way: in objects

$$\begin{array}{rcl} q = \coprod_{x\in C} u^x\colon \coprod_{x\in C} I_x & \longrightarrow & \mathrm{Obj}(D) \\ i & \longmapsto & u^x_i, \text{ if } i \in I_x. \end{array}$$

Now observe that for every $z_1, z_2 \in M$ such that $\mathrm{Hom}_M(z_1, z_2) \neq \varnothing$, we also have that

$$\mathrm{Hom}_D(q(z_1), q(z_2)) \neq \varnothing$$

is non-empty by construction. Indeed, in such a case there exists $x \sim y$ in C and a bijection $\varphi_{x,y}\colon I_x \longrightarrow I_y$, with $\varphi_{x,y}(z_1) = z_2$, such that

$$q(i) = u^x_i \sim u^y_{\varphi_{x,y}(i)} = q(\varphi_{x,y}(i))$$

for every $i \in I_x$, and in particular $q(z_1) \sim q(z_2)$. Therefore, it makes sense to define

$$\begin{array}{rcl} q = q_{z_1,z_2}\colon \mathrm{Hom}_M(z_1, z_2) & \longrightarrow & \mathrm{Hom}_D(q(z_1), q(z_2)) \\ a^M_{z_1,z_2} & \longmapsto & a^D_{q(z_1),q(z_2)}. \end{array}$$

Now we just need to consider the isomorphism class of the span given by

$$\begin{tikzcd} C \arrow[d, "D_C"'] & M \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "D_M"] & D \arrow[d, "D_D"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}, \end{tikzcd}$$

where D_M is the unique morphism that verifies

$$D_C \circ p = D_M = D_D \circ q,$$

whose existence is guaranteed by the fact that f is a cell-map, that is, the equality

$$D_C \circ p = D_D \circ q$$

holds precisely because f preserves dimensions. Then, for every $x \in \Psi(C)$,

$$\begin{tikzcd} \{*\} \arrow[d, "\overline{x}"'] & I_x \arrow[l, "\widetilde{p}"'] \arrow[d, "\widetilde{\overline{x}}"] \arrow[dl, phantom, "\lrcorner", very near start] \arrow[dr, dashed, red, bend left, "\Psi(p,q)(x)=(u_i^x)_{i\in I_x}"] & \\ \Psi(C) & \Psi(M) \arrow[l, "p"] \arrow[r, "q"'] & \Psi(D). \end{tikzcd}$$

In equations,

$$\Psi(p,q)(x) = (q(m))_{m\in p^{-1}(x)} = (q(i))_{i\in I_x} = (u_i^x)_{i\in I_x}.$$

Thus, $\Psi(p,q) = f$, so Ψ is surjective on arrows. ∎

Observe that the faithfulness of Ψ fails precisely because of the freedom we have to construct the groupoid M. In our construction, within each fiber we only put the identities, and in general we could add more arrows in M such that the hole construction makes sense. For instance, consider the cell-map $f\colon \mathscr{T} \longrightarrow \mathscr{T}$ defined as

$$\bullet \xmapsto{\ f\ } \bullet^2,$$

that is, it maps the unique element $\bullet$ to itself $\bullet^2$ with multiplicity 2. Following the construction we made in the proof of 3.2.12, the preimage of f under Ψ corresponds to the span

$$\mathscr{T} \xleftarrow{\ p\ } M \xrightarrow{\ q\ } \mathscr{T},$$

where M is the discrete groupoid given by

$$M := I_\bullet = \begin{matrix} {\color{red}\bullet} \\ \\ {\color{blue}\bullet} \end{matrix}$$

and the span is defined as

$$\bullet \xleftarrow{\ p\ }\!\!\!\shortmid \quad \begin{matrix} {\color{red}\bullet} \\ \\ {\color{blue}\bullet} \end{matrix} \quad \shortmid\!\!\!\xrightarrow{\ q\ } \bullet.$$

Hence, we have that

$$\Psi(p,q)(\bullet) = (q(m))_{m\in\{{\color{red}\bullet},{\color{blue}\bullet}\}} = \bullet^2.$$

This small example illustrates well the problem with the faithfulness: observe that we can add an extra arrow in the fiber $I_\bullet$ without changing the result under Ψ. That is, consider the new groupoid

$$M' := I'_\bullet = \begin{tikzcd} {\color{red}\bullet} \arrow[d, phantom, "\wr"] \\ {\color{blue}\bullet} \end{tikzcd}$$

and the span

$$\mathscr{T} \xleftarrow{p'} M' \xrightarrow{q'} \mathscr{T}$$

defined as

$$\bullet \xleftarrow{\ p'\ } \begin{matrix} {\color{red}\bullet} \\ \wr \\ {\color{blue}\bullet} \end{matrix} \xmapsto{\ q'\ } \bullet.$$

Here, p' and q' are clearly functors, and the p-property is not violated. Moreover,

$$\Psi(p',q')(\bullet) = (q'(m))_{m\in\{{\color{red}\bullet},{\color{blue}\bullet}\}} = \bullet^2.$$

Thus, we have constructed two non-isomorphic spans that give the same cell-map under Ψ, showing the failure of the faithfulness.

3.3 Monoidal Structure in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$

Finally, we construct the desired monoidal structure on our category of spans. Consider the assignment

$$\otimes\colon \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \times \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \longrightarrow \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$$

defined as follows: on objects, it acts as

$$\left(\begin{tikzcd} A \arrow[d, "D_A"] \\ \mathbb{Z} \end{tikzcd}, \begin{tikzcd} B \arrow[d, "D_B"] \\ \mathbb{Z} \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} A \arrow[d, "D_A"] \\ \mathbb{Z} \end{tikzcd}\right) \otimes \left(\begin{tikzcd} B \arrow[d, "D_B"] \\ \mathbb{Z} \end{tikzcd}\right) := \begin{tikzcd} A\times B \arrow[d, "D_A\times D_B"] \\ \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z}, \end{tikzcd}$$

and in arrows we define it as

$$\left(\begin{tikzcd} A_1 \arrow[d, "D_{A_1}"'] & M \arrow[l, "x"'] \arrow[r, "y"] \arrow[d, "D_M"'] & B_1 \arrow[d, "D_{B_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \otimes \left(\begin{tikzcd} A_2 \arrow[d, "D_{A_2}"'] & N \arrow[l, "z"'] \arrow[r, "t"] \arrow[d, "D_N"'] & B_2 \arrow[d, "D_{B_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) :=$$

$$
\begin{tikzcd}
A_1 \times A_2 \arrow[d, "D_{A_1}+D_{A_2}"'] & M \times N \arrow[l, "x\times z"'] \arrow[r, "y\times t"] \arrow[d, "D_M+D_N"'] & B_1 \times B_2 \arrow[d, "D_{B_1}+D_{B_2}"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}.
\end{tikzcd}
\tag{3.15}
$$

First of all, we have to check that such an assignment is well-defined. Of course, on objects everything works fine. On arrows, we clearly have that if the diagrams

$$
\begin{tikzcd}
A_1 \arrow[d, "D_{A_1}"'] & M \arrow[l, "x"'] \arrow[r, "y"] \arrow[d, "D_M"'] & B_1 \arrow[d, "D_{B_1}"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[d, "D_{A_2}"'] & N \arrow[l, "z"'] \arrow[r, "t"] \arrow[d, "D_N"'] & B_2 \arrow[d, "D_{B_2}"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}
$$

are spans in $\mathbf{DiscGrpd}/\mathbb{Z}$, then so is the product

$$
\begin{tikzcd}
A_1 \times A_2 \arrow[d, "D_{A_1}+D_{A_2}"'] & M \times N \arrow[l, "x\times z"'] \arrow[r, "y\times t"] \arrow[d, "D_M+D_N"'] & B_1 \times B_2 \arrow[d, "D_{B_1}+D_{B_2}"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}.
\end{tikzcd}
$$

Thus, the only thing that remains to be checked is that $\otimes$ is well-defined on isomorphism classes, and the following lemma shows that this is indeed true.

Lemma 3.3.1. *Consider the following pairs of isomorphic spans of discrete groupoids over* $\mathbb{Z}$

$$
\begin{tikzcd}
 & A \arrow[dr, "\varphi_1"] \arrow[dd, "D_A"] & & P \arrow[ll, "c"'] \arrow[dlll] \arrow[dl, "d"] \arrow[dd, "D_P"] \\
R \arrow[ur, "a"] \arrow[rr, "b"] \arrow[dd, "D_R"'] & & C \arrow[dd, "D_C"] & \\
 & \mathbb{Z} \arrow[rr, equal] \arrow[dl, equal] & & \mathbb{Z} \arrow[dl, equal] \arrow[dlll, equal] \\
\mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A' \arrow[dr, "\varphi_2"] \arrow[dd, "D_{A'}"] & & P' \arrow[ll, "c'"'] \arrow[dlll] \arrow[dl, "d'"] \arrow[dd, "D_{P'}"] \\
R' \arrow[ur, "a'"] \arrow[rr, "b'"] \arrow[dd, "D_{R'}"'] & & C' \arrow[dd, "D_{C'}"] & \\
 & \mathbb{Z} \arrow[rr, equal] \arrow[dl, equal] & & \mathbb{Z} \arrow[dl, equal] \arrow[dlll, equal] \\
\mathbb{Z} \arrow[rr, equal] & & \mathbb{Z}. &
\end{tikzcd}
\tag{3.16}
$$

Then,

$$
\begin{tikzcd}
 & A \times A' \arrow[dr, "\varphi_1\times\varphi_2"] \arrow[dd, "D_A+D_{A'}"] & & P \times P' \arrow[ll, "c\times c'"'] \arrow[dlll] \arrow[dl, "d\times d'"] \arrow[dd, "D_P+D_{P'}"] \\
R \times R' \arrow[ur, "a\times a'"] \arrow[rr, "b\times b'"] \arrow[dd, "D_R+D_{R'}"'] & & C \times C' \arrow[dd, "D_C+D_{C'}"] & \\
 & \mathbb{Z} \arrow[rr, equal] \arrow[dl, equal] & & \mathbb{Z} \arrow[dl, equal] \arrow[dlll, equal] \\
\mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} &
\end{tikzcd}
\tag{3.17}
$$

is also an isomorphism of spans.

Proof. Clearly if φ_1 and φ_2 are isomorphisms, then so is $\varphi_1 \times \varphi_2$, just by functoriality of the product $\times$. Furthermore, the top face of the square 3.17

$$\begin{tikzcd}
 & P \times P' \arrow[dl, "c\times c'"'] \arrow[dr, "d\times d'"] \arrow[dd, "\varphi_1\times\varphi_2"'] & \\
A \times A' & & C \times C' \\
 & R \times R' \arrow[ul, "a\times a'"] \arrow[ur, "b\times b'"'] &
\end{tikzcd}$$

is commutative, since the top faces of the squares in 3.16 commute. Hence, it only remains to see that the inner diagonal rectangle in 3.17

$$\begin{tikzcd}
P \times P' \arrow[r, "\varphi_1\times\varphi_2"] \arrow[d, "D_P+D_{P'}"'] & R \times R' \arrow[d, "D_R+D_{R'}"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}$$

is commutative, and this is again easy to see: we just need to extract the following equations from 3.16

$$\begin{cases} D_R \circ \varphi_1 = D_P \\ D_{R'} \circ \varphi_2 = D_{P'}. \end{cases}$$

If we add them, we get that

$$(D_R + D_{R'}) \circ (\varphi_1 \times \varphi_2) = D_R \circ \varphi_1 + D_{R'} \circ \varphi_2 = D_P + D_{P'},$$

as wanted. ■

Before showing that the above assignment gives a monoidal structure, we give a lemma that will be useful for such a purpose. We do not provide a proof, since it is analogous to the one made in 3.3.1.

Lemma 3.3.2. *Consider the following pair of diagrams in* $\textbf{DiscGrpd}/\mathbb{Z}$

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
P \arrow[ur, "D_P"] \arrow[rr, "q"] \arrow[dd, "p"'] & & C \arrow[ur, "D_C"] \arrow[dd, "g"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
A \arrow[ur, "D_A"] \arrow[rr, "f"] & & B \arrow[ur, "D_B"] &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
P' \arrow[ur, "D_{P'}"] \arrow[rr, "q'"] \arrow[dd, "p'"'] & & C' \arrow[ur, "D_{C'}"] \arrow[dd, "g'"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
A' \arrow[ur, "D_{A'}"] \arrow[rr, "f'"] & & B' \arrow[ur, "D_{B'}"] &
\end{tikzcd}
\tag{3.18}$$

and assume that they are pullbacks in $\textbf{DiscGrpd}/\mathbb{Z}$*. Then, so is the cube*

$$
\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
P \times P' \arrow[ur, "D_P+D_{P'}"] \arrow[rr, "q\times q'"] \arrow[dd, "p\times p'"'] & & C \times C' \arrow[ur, "D_C+D_{C'}"] \arrow[dd, "g\times g'"] & \\
& \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
A \times A' \arrow[ur, "D_A+D_{A'}"] \arrow[rr, "f\times f'"] & & B \times B'. \arrow[ur, "D_B+D_{B'}"'] &
\end{tikzcd}
\tag{3.19}
$$

Equivalently, the span

$$
\begin{tikzcd}
A \times A' \arrow[d, "D_A+D_A"'] & P \times P' \arrow[l, "p\times p'"'] \arrow[r, "q\times q'"] \arrow[d, "D_P+D_{P'}"'] & C \times C' \arrow[d, "D_C+D_{C'}"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}
\tag{3.20}
$$

is isomorphic in the bicategory of spans $\mathbf{Span}(\mathbf{DiscGrpd}/\mathbb{Z})$ *to some pullback of the cospan*

$$
\begin{tikzcd}
A \times A' \arrow[r, "f\times f'"] \arrow[d, "D_A+D_A"'] & B \times B' \arrow[d, "D_P+D_{P'}"'] & C \times C' \arrow[l, "g\times g'"'] \arrow[d, "D_C+D_{C'}"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}.
\end{tikzcd}
\tag{3.21}
$$

Now we are ready to state and prove the desired result.

Proposition 3.3.3. *The above-defined assignment*

$$\otimes\colon \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \times \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \longrightarrow \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$$

is a functor that turns $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$ *into a symmetric monoidal category, where the rest of the structure is essentially given by the symmetric monoidal structure of*

$$(\mathbf{DiscGrpd}, \times, 1, B),$$

that is, the unit is given by the terminal groupoid together with the zero-dimension functor

$$
I = \begin{tikzcd} 1 \arrow[d, "0"] \\ \mathbb{Z}, \end{tikzcd}
$$

the associator is the span

$$a = a_{A,B,C} = \begin{tikzcd} (A\times B)\times C \arrow[r, equal] \arrow[d, "(D_A+D_B)\times D_C"'] & (A\times B)\times C \arrow[r, "a_{A,B,C}"] \arrow[d, "(D_A+D_B)\times D_C"'] & A\times(B\times C) \arrow[d, "D_A\times(D_B+D_C)"'] \\ \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}, \end{tikzcd}$$

the left unitor is defined as

$$\lambda = \lambda_A = \begin{tikzcd} 1\times A \arrow[r, equal] \arrow[d, "0+D_A"'] & 1\times A \arrow[r, "\lambda_A"] \arrow[d, "0+D_A"'] & A \arrow[d, "D_A"'] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}, \end{tikzcd}$$

the right unitor is defined analogously as

$$\rho = \rho_A = \begin{tikzcd} A\times 1 \arrow[r, equal] \arrow[d, "D_A+0"'] & A\times 1 \arrow[r, "\rho_A"] \arrow[d, "D_A+0"'] & A \arrow[d, "D_A"'] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

and the braiding is determined by

$$B = B_{A,C} = \begin{tikzcd} A\times C \arrow[r, equal] \arrow[d, "D_A+D_C"'] & A\times C \arrow[r, "B_{A,C}"] \arrow[d, "D_A+D_C"'] & C\times A \arrow[d, "D_C+D_A"'] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Proof. Let us first prove functoriality. Write $F := \otimes$, so that if

$$\begin{tikzcd} A \arrow[d, "D_A"] \\ \mathbb{Z} \end{tikzcd}, \quad \begin{tikzcd} A \arrow[d, "D_A"] \\ \mathbb{Z} \end{tikzcd} \in \textbf{DiscGrpd}/\mathbb{Z},$$

then

$$F(A,B) = A\otimes B,$$

omitting the dimension functors. Notation for the spans is analogously defined. As always, identities are always easily respected:

$$F(\mathrm{id}_{(A,B)}) = F(\mathrm{id}_A, \mathrm{id}_B) = \begin{tikzcd} A\times B \arrow[d, "D_A+D_B"'] & A\times B \arrow[l, "\mathrm{id}_A\times\mathrm{id}_B"'] \arrow[r, "\mathrm{id}_A\times\mathrm{id}_B"] \arrow[d, "D_A+D_B"'] & A\times B \arrow[d, "D_A+D_B"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd} =$$

$$\begin{tikzcd} A\times B \arrow[r, equal] \arrow[d, "D_A+D_B"'] & A\times B \arrow[r, equal] \arrow[d, "D_A+D_B"'] & A\times B \arrow[d, "D_A+D_B"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd} = \mathrm{id}_{F(A,B)},$$

since $\mathrm{id}_A \times \mathrm{id}_B = \mathrm{id}_{A\times B}$ just by functoriality of the product $\times$. To see that F respects composition is a little harder. Consider the sequence of arrows

$$\left(\begin{tikzcd} A_1 \arrow[d, "D_{A_1}"] \\ \mathbb{Z} \end{tikzcd}, \begin{tikzcd} A_2 \arrow[d, "D_{A_2}"] \\ \mathbb{Z} \end{tikzcd}\right) \xrightarrow{(S_1,S_2)} \left(\begin{tikzcd} B_1 \arrow[d, "D_{B_1}"] \\ \mathbb{Z} \end{tikzcd}, \begin{tikzcd} B_2 \arrow[d, "D_{B_2}"] \\ \mathbb{Z} \end{tikzcd}\right) \xrightarrow{(S_3,S_4)} \left(\begin{tikzcd} C_1 \arrow[d, "D_{C_1}"] \\ \mathbb{Z} \end{tikzcd}, \begin{tikzcd} C_2 \arrow[d, "D_{C_2}"] \\ \mathbb{Z} \end{tikzcd}\right),$$

where S_1, S_2, S_3 and S_4 are spans given by

$$S_1 = (x,y) = \begin{tikzcd} A_1 \arrow[d, "D_{A_1}"'] & M \arrow[l, "x"'] \arrow[r, "y"] \arrow[d, "D_M"'] & B_1 \arrow[d, "D_{B_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}, \qquad S_2 = (z,t) = \begin{tikzcd} A_2 \arrow[d, "D_{A_2}"'] & N \arrow[l, "z"'] \arrow[r, "t"] \arrow[d, "D_N"'] & B_2 \arrow[d, "D_{B_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd},$$

$$S_3 = (x',y') = \begin{tikzcd} B_1 \arrow[d, "D_{B_1}"'] & M' \arrow[l, "x'"'] \arrow[r, "y'"] \arrow[d, "D_{M'}"'] & C_1 \arrow[d, "D_{C_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}, \qquad S_4 = (z',t') = \begin{tikzcd} B_2 \arrow[d, "D_{B_2}"'] & N' \arrow[l, "z'"'] \arrow[r, "t'"] \arrow[d, "D_{N'}"'] & C_2 \arrow[d, "D_{C_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}.$$

What we want to show is that

$$\Big[F((S_3,S_4)\circ(S_1,S_2))\Big] = \Big[F(S_3,S_4)\circ F(S_1,S_2)\Big] \tag{3.22}$$

in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$. On the one hand, we have that

$$F((S_3,S_4)\circ(S_1,S_2)) = F(S_3\circ S_1, S_4\circ S_2).$$

Using proposition 3.2.5, we know that the composite $S_3 \circ S_1$ is equal to the blue part of the following diagram

$$\begin{tikzcd} & & M\times_{B_1} M' \arrow[dl, "\widetilde{x'}"'] \arrow[dr, "\widetilde{y}"] \arrow[d, "D_{M\times_{B_1}M'}"] & & \\ A_1 \arrow[d, "D_{A_1}"'] & M \arrow[l, "x"'] \arrow[d, "D_M"'] \arrow[dr, "y"] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & M' \arrow[r, "y'"] \arrow[d, "D_{M'}"] \arrow[dl, "x'"'] & C_1 \arrow[d, "D_{C_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[dr, equal] & B_1 \arrow[d, "D_{B_1}"] & \mathbb{Z} \arrow[r, equal] \arrow[dl, equal] & \mathbb{Z}, \\ & & \mathbb{Z} & & \end{tikzcd}$$

that is,

$$S_3 \circ S_1 = \begin{tikzcd} A_1 \arrow[d, "D_{A_1}"'] & M \times_{B_1} M' \arrow[l, "x\circ\widetilde{x}'"'] \arrow[r, "y'\circ\widetilde{y}"] \arrow[d, "D_{M\times_{B_1}M'}"] & C_1 \arrow[d, "D_{C_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Again, using proposition 3.2.5, we deduce that the composite $S_4 \circ S_2$ equals

$$\begin{tikzcd} & & N \times_{B_2} N' \arrow[dl, "\widetilde{z}'"'] \arrow[d, "D_{N\times_{B_2}N'}"] \arrow[dr, "\widetilde{t}"] & & \\ A_2 \arrow[d, "D_{A_2}"'] & N \arrow[l, "z"'] \arrow[d, "D_N"'] \arrow[dr, "t"] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & N' \arrow[r, "t'"] \arrow[d, "D_{N'}"] \arrow[dl, "z'"'] & C_2 \arrow[d, "D_{C_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[dr, equal] & B_2 \arrow[d, "D_{B_2}"] & \mathbb{Z} \arrow[r, equal] \arrow[dl, equal] & \mathbb{Z} \\ & & \mathbb{Z} & & \end{tikzcd}$$

which, written in a more compact way, means that

$$S_4 \circ S_2 = \begin{tikzcd} A_2 \arrow[d, "D_{A_2}"'] & N \times_{B_2} N' \arrow[l, "z\circ\widetilde{z}'"'] \arrow[r, "t'\circ\widetilde{t}"] \arrow[d, "D_{N\times_{B_2}N'}"] & C_2 \arrow[d, "D_{C_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Thus, we have that

$$F((S_3, S_4) \circ (S_1, S_2)) = F(S_3 \circ S_1, S_4 \circ S_2) =$$

$$\begin{tikzcd} A_1 \times A_2 \arrow[d, "D_{A_1}+D_{A_2}"'] & (M \times_{B_1} M') \times (N \times_{B_2} N') \arrow[l, "(x\circ\widetilde{x}')\times(z\circ\widetilde{z}')"'] \arrow[r, "(y'\circ\widetilde{y})\times(t'\circ\widetilde{t})"] \arrow[d, "D_{M\times_{B_1}M'}+D_{N\times_{B_2}N'}"] & C_1 \times C_2 \arrow[d, "D_{C_1}+D_{C_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd} \tag{3.23}$$

Let us now compute the right hand side of 3.22. We have that

$$F(S_3, S_4) \circ F(S_1, S_2) =$$

$$\left(\begin{tikzcd} B_1 \times B_2 \arrow[d, "D_{B_1}+D_{B_2}"] & M' \times N' \arrow[l, "x'\times z'"'] \arrow[r, "y'\times t'"] \arrow[d, "D_{M'}+D_{N'}"] & C_1 \times C_2 \arrow[d, "D_{C_1}+D_{C_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \circ \left(\begin{tikzcd} A_1 \times A_2 \arrow[d, "D_{A_1}+D_{A_2}"] & M \times N \arrow[l, "x\times z"'] \arrow[r, "y\times t"] \arrow[d, "D_M+D_N"] & B_1 \times B_2 \arrow[d, "D_{B_1}+D_{B_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) =$$

$$
\begin{tikzcd}
 & & & (M\times N)\times_{B_1\times B_2}(M'\times N') \arrow[dll, "\widetilde{x'\times z'}"'] \arrow[d] \arrow[drr, "\widetilde{y\times t}"] & & & \\
A_1\times A_2 \arrow[d] & M\times N \arrow[l, "x\times z"'] \arrow[d] \arrow[drr, "y\times t"] & & \mathbb{Z} \arrow[dll, equal] \arrow[drr, equal] & & M'\times N' \arrow[r, "y'\times t'"] \arrow[d] \arrow[dll, "x'\times z'"'] & C_1\times C_2 \arrow[d] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[drr, equal] & & B_1\times B_2 \arrow[d] & & \mathbb{Z} \arrow[r, equal] \arrow[dll, equal] & \mathbb{Z} \\
 & & & \mathbb{Z} & & &
\end{tikzcd}
=
$$

$$
\begin{tikzcd}
A_1\times A_2 \arrow[d, "D_{A_1}+D_{A_2}"'] & (M\times N)\times_{B_1\times B_2}(M'\times N') \arrow[l, "(x\times z)\circ(\widetilde{x'\times z'})"'] \arrow[r, "(y'\times t')\circ(\widetilde{y\times t})"] \arrow[d, "D_{(M\times N)\times_{B_1\times B_2}(M'\times N')}"] & C_1\times C_2 \arrow[d, "D_{C_1}+D_{C_2}"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}.
\end{tikzcd}
\tag{3.24}
$$

Now we just have to show that the spans 3.23 and 3.24 are isomorphic as spans of discrete groupoids. Observe that, since we have this pair of pullback squares

$$
\begin{tikzcd}
 & M\times_{B_1}M' \arrow[dl, "\widetilde{x'}"'] \arrow[d, "D_{M\times_{B_1}M'}"] \arrow[dr, "\widetilde{y}"] & \\
M \arrow[d, "D_M"'] \arrow[dr, "y"] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & M' \arrow[d, "D_{M'}"] \arrow[dl, "x'"'] \\
\mathbb{Z} \arrow[dr, equal] & B_1 \arrow[d, "D_{B_1}"] & \mathbb{Z} \arrow[dl, equal] \\
 & \mathbb{Z} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N\times_{B_2}N' \arrow[dl, "\widetilde{z'}"'] \arrow[d, "D_{N\times_{B_2}N'}"] \arrow[dr, "\widetilde{t}"] & \\
N \arrow[d, "D_N"'] \arrow[dr, "t"] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & N' \arrow[d, "D_{N'}"] \arrow[dl, "z'"'] \\
\mathbb{Z} \arrow[dr, equal] & B_2 \arrow[d, "D_{B_2}"] & \mathbb{Z} \arrow[dl, equal] \\
 & \mathbb{Z}, &
\end{tikzcd}
$$

from lemma 3.3.2 we have that the square

$$
\begin{tikzcd}
 & (M\times_{B_1}M')\times(N\times_{B_2}N') \arrow[dl, "\widetilde{x'}\times\widetilde{z'}"'] \arrow[d] \arrow[dr, "\widetilde{y}\times\widetilde{t}"] & \\
M\times N \arrow[d] \arrow[dr, "y\times t"] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & M'\times N' \arrow[d] \arrow[dl, "x'\times z'"'] \\
\mathbb{Z} \arrow[dr, equal] & B_1\times B_2 \arrow[d] & \mathbb{Z} \arrow[dl, equal] \\
 & \mathbb{Z} &
\end{tikzcd}
$$

is a pullback as well. Therefore, there exists an isomorphism of spans

$$
\begin{tikzcd}
& (M\times_{B_1}M')\times(N\times_{B_2}N') \arrow[dl, "\widetilde{x'}\times\widetilde{z'}"'] \arrow[d] \arrow[dr, "\widetilde{y}\times\widetilde{t}"] \arrow[dd, bend left, "\varphi"] & \\
M\times N \arrow[d] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, equal] & M'\times N' \arrow[d] \\
\mathbb{Z} \arrow[dr, equal] & (M\times N)\times_{B_1\times B_2}(M'\times N') \arrow[ul, "\widetilde{x'\times z'}"] \arrow[ur, "\widetilde{y\times t}"'] \arrow[d] \arrow[d, bend left, equal] & \mathbb{Z}. \arrow[dl, equal] \\
& \mathbb{Z} &
\end{tikzcd}
$$

Consider now the above diagram extended in the following way

$$
\begin{tikzcd}
& & (M\times_{B_1}M')\times(N\times_{B_2}N') \arrow[dll, bend right, "(x\circ\widetilde{x'})\times(z\circ\widetilde{z'})"'] \arrow[drr, bend left, "(y'\circ\widetilde{y})\times(t'\circ\widetilde{t})"] \arrow[dl, "\widetilde{x'}\times\widetilde{z'}"'] \arrow[d] \arrow[dr, "\widetilde{y}\times\widetilde{t}"] \arrow[dd, bend left, "\varphi"] & & \\
A_1\times A_2 \arrow[d] & M\times N \arrow[l, "x\times z"'] \arrow[d] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] \arrow[dll, bend right, equal] \arrow[drr, bend left, equal] \arrow[dd, equal] & M'\times N' \arrow[r, "y'\times t'"] \arrow[d] & C_1\times C_2 \arrow[d] \\
\mathbb{Z} \arrow[r, equal] \arrow[drr, bend right, equal] & \mathbb{Z} \arrow[dr, equal] & (M\times N)\times_{B_1\times B_2}(M'\times N') \arrow[ul, "\widetilde{x'\times z'}"] \arrow[ur, "\widetilde{y\times t}"'] \arrow[ull, bend left, "(x\times z)\circ(\widetilde{x'\times z'})"] \arrow[urr, bend right, "(y'\times t')\circ(\widetilde{y\times t})"'] \arrow[d] \arrow[d, bend left, equal] & \mathbb{Z} \arrow[dl, equal] \arrow[r, equal] & \mathbb{Z}. \arrow[dll, bend left, equal] \\
& & \mathbb{Z} & &
\end{tikzcd}
$$

We observe that this diagram is an isomorphism of spans between 3.23 and 3.24 if and only if the four outer triangles commute. The pair at the bottom trivially commutes, and the pair at the top commutes if and only if

$$
\begin{cases}
(x\times z)\circ(\widetilde{x'}\times\widetilde{z'}) = (x\circ\widetilde{x'})\times(z\circ\widetilde{z'}),\\
(y'\times t')\circ(\widetilde{y}\times\widetilde{t}) = (y'\circ\widetilde{y})\times(t'\circ\widetilde{t}).
\end{cases}
$$

But these equalities trivially hold because of the functoriality of the cartesian product. Therefore, the spans 3.23 and 3.24 are isomorphic, which means that

$$
F((S_3,S_4)\circ(S_1,S_2)) \cong F(S_3,S_4)\circ F(S_1,S_2),
$$

as desired. Thus, $\otimes$ is a functor. The associator, the left unitor, the right unitor and the braiding defined on $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$ are easily seen to verify the axioms as a consequence of the symmetric monoidal structure of **DiscGrpd**. For instance, let us check the triangle identity

$$
\begin{tikzcd}
(A\otimes 1)\otimes B \arrow[rr, "a_{A,1,B}"] \arrow[dr, "\rho_A\otimes \mathrm{id}_B"'] & & A\otimes(1\otimes B) \arrow[dl, "\mathrm{id}_A\otimes\lambda_B"] \\
& A\otimes B. &
\end{tikzcd}
$$

Expanding the above diagram (and omitting the dimension functors)

$$\begin{tikzcd}
(A\times 1)\times B \arrow[r, equal] \arrow[dr, equal] & (A\times 1)\times B \arrow[r, "a_{A,1,B}"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & A\times(1\times B) \arrow[d, equal] \\
& (A\times 1)\times B \arrow[r, "a_{A,1,B}"] \arrow[dr, "\rho_A\times \mathrm{id}_B"'] & A\times(1\times B) \arrow[d, "\mathrm{id}_A\times\lambda_B"] \\
& & A\times B,
\end{tikzcd}$$

we see that the diagram of spans commutes if and only if the bottom triangle commutes, which does so thanks to the monoidal structure of **DiscGrpd**. ■

Proposition 3.3.4. *The Cellularization functor*

$$\Psi\colon (\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}), \otimes, I, B) \longrightarrow (\mathbf{Cell}, \times, \mathscr{T}, \tau)$$

is a strict monoidal functor.

Proof. Let us first check that

$$\Psi\left(\begin{pmatrix} A \\ \downarrow D_A \\ \mathbb{Z}\end{pmatrix} \otimes \begin{pmatrix} B \\ \downarrow D_B \\ \mathbb{Z}\end{pmatrix}\right) = \Psi\begin{pmatrix} A \\ \downarrow D_A \\ \mathbb{Z}\end{pmatrix} \times \Psi\begin{pmatrix} B \\ \downarrow D_B \\ \mathbb{Z}\end{pmatrix}. \tag{3.25}$$

On the one hand, the cell-set

$$\Psi\left(\begin{pmatrix} A \\ \downarrow D_A \\ \mathbb{Z}\end{pmatrix} \otimes \begin{pmatrix} B \\ \downarrow D_B \\ \mathbb{Z}\end{pmatrix}\right) = \Psi\begin{pmatrix} A\times B \\ \downarrow D_A+D_B \\ \mathbb{Z}\end{pmatrix}$$

is defined as follows: it has $\mathrm{Obj}(A\times B) = \mathrm{Obj}(A)\times\mathrm{Obj}(B)$ as underlying set. The equivalence relation on $\mathrm{Obj}(A\times B)$ is given by

$$(a,b)\sim(a',b') \iff |\operatorname{Hom}_A(a,a')||\operatorname{Hom}_B(b,b')| = |\operatorname{Hom}_{A\times B}((a,b),(a',b'))| = 1,$$

which is the same as saying that

$$(a,b)\sim(a',b') \iff |\operatorname{Hom}_A(a,a')| = |\operatorname{Hom}_B(b,b')| = 1 \iff a\sim a' \text{ and } b\sim b',$$

that is, the equivalence relation is defined componentwise. Finally, the dimension function is just given by the sum of the functors D_A and D_B restricted on objects. This cell-set is exactly the same as the one in the right hand side of 3.25. Now in order to see that the identity

$$\mathrm{id} = \Psi_{A,B}\colon \Psi\left(\begin{pmatrix} A \\ \downarrow D_A \\ \mathbb{Z}\end{pmatrix} \otimes \begin{pmatrix} B \\ \downarrow D_B \\ \mathbb{Z}\end{pmatrix}\right) \longrightarrow \Psi\begin{pmatrix} A \\ \downarrow D_A \\ \mathbb{Z}\end{pmatrix} \times \Psi\begin{pmatrix} B \\ \downarrow D_B \\ \mathbb{Z}\end{pmatrix}$$

is a structure map, we need to check that it defines a natural transformation, that is, we need to check that

$$\Psi\left(\left(\begin{tikzcd} A_1 \arrow[d, "D_{A_1}"'] & M \arrow[l, "x"'] \arrow[r, "y"] \arrow[d, "D_M"'] & B_1 \arrow[d, "D_{B_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \otimes \left(\begin{tikzcd} A_2 \arrow[d, "D_{A_2}"'] & N \arrow[l, "z"'] \arrow[r, "t"] \arrow[d, "D_N"'] & B_2 \arrow[d, "D_{B_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right)\right) =$$
$$\Psi\left(\begin{tikzcd} A_1 \arrow[d, "D_{A_1}"'] & M \arrow[l, "x"'] \arrow[r, "y"] \arrow[d, "D_M"'] & B_1 \arrow[d, "D_{B_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \times \Psi\left(\begin{tikzcd} A_2 \arrow[d, "D_{A_2}"'] & N \arrow[l, "z"'] \arrow[r, "t"] \arrow[d, "D_N"'] & B_2 \arrow[d, "D_{B_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right). \tag{3.26}$$

On the one hand, the left-hand side of 3.26 is

$$\Psi\left(\left(\begin{tikzcd} A_1 \arrow[d, "D_{A_1}"'] & M \arrow[l, "x"'] \arrow[r, "y"] \arrow[d, "D_M"'] & B_1 \arrow[d, "D_{B_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \otimes \left(\begin{tikzcd} A_2 \arrow[d, "D_{A_2}"'] & N \arrow[l, "z"'] \arrow[r, "t"] \arrow[d, "D_N"'] & B_2 \arrow[d, "D_{B_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right)\right) =$$

$$\Psi\left(\begin{tikzcd} A_1 \times A_2 \arrow[d, "D_{A_1}+D_{A_2}"'] & M \times N \arrow[l, "x\times z"'] \arrow[r, "y\times t"] \arrow[d, "D_M+D_N"'] & B_1 \times B_2 \arrow[d, "D_{B_1}+D_{B_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) =$$

$$\begin{aligned} \Psi(x \times z, y \times t)\colon \Psi(A_1) \times \Psi(A_2) &\longrightarrow \Psi(B_1) \times \Psi(B_2) \\ (a_1, a_2) &\longmapsto ((y(m), t(n)))_{(m,n) \in x^{-1}(a_1) \times z^{-1}(a_2)}, \end{aligned}$$

On the other hand, the right hand side of 3.26 is

$$\Psi\left(\begin{tikzcd} A_1 \arrow[d, "D_{A_1}"'] & M \arrow[l, "x"'] \arrow[r, "y"] \arrow[d, "D_M"'] & B_1 \arrow[d, "D_{B_1}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \times \Psi\left(\begin{tikzcd} A_2 \arrow[d, "D_{A_2}"'] & N \arrow[l, "z"'] \arrow[r, "t"] \arrow[d, "D_N"'] & B_2 \arrow[d, "D_{B_2}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) =$$

$$\begin{aligned} \Psi(x, y) \times \Psi(z, t)\colon \Psi(A_1) \times \Psi(A_2) &\longrightarrow \Psi(B_1) \times \Psi(B_2) \\ (a_1, a_2) &\longmapsto (\Psi(x, y) \times \Psi(z, t))(a_1, a_2) \end{aligned}$$

where, for every $(a_1, a_2) \in \Psi(A_1) \times \Psi(A_2)$,

$$(\Psi(x, y) \times \Psi(z, t))(a_1, a_2) = \Psi(x, y)(a_1) \times \Psi(z, t)(a_2) =$$

$$(y(m))_{m \in x^{-1}(a_1)} \times (t(n))_{n \in z^{-1}(a_2)} = ((y(m), t(n)))_{(m,n) \in x^{-1}(a_1) \times z^{-1}(a_2)}$$

Thus, both sides of 3.26 coincide and therefore the identity

$$\mathrm{id} = \Psi_{A,B}\colon \Psi\left(\left(\begin{tikzcd} A \arrow[d, "D_A"] \\ \mathbb{Z} \end{tikzcd}\right) \otimes \left(\begin{tikzcd} B \arrow[d, "D_B"] \\ \mathbb{Z} \end{tikzcd}\right)\right) \longrightarrow \Psi\left(\begin{tikzcd} A \arrow[d, "D_A"] \\ \mathbb{Z} \end{tikzcd}\right) \times \Psi\left(\begin{tikzcd} B \arrow[d, "D_B"] \\ \mathbb{Z} \end{tikzcd}\right)$$

is a structure map. Moreover, it is clear that

$$\Psi\left(\begin{array}{c} 1 \\ \big\downarrow{\scriptstyle 0} \\ \mathbb{Z} \end{array}\right) = \mathscr{T},$$

hence the second structure map is also the identity. Since the structure maps are identities, the rest of the compatibility conditions are equivalent to the equalities

$$\Psi(a_{A,B,C}) = a_{\Psi(A),\Psi(B),\Psi(C)},\ \Psi(\lambda_A) = \lambda_{\Psi(A)} \text{ and } \Psi(\rho_A) = \rho_{\Psi(A)},$$

which trivially hold. For instance,

$$\Psi(a_{A,B,C}) = \Psi\left(\begin{tikzcd} (A\times B)\times C \arrow[r, equal] \arrow[d, "(D_A+D_B)+D_C"'] & (A\times B)\times C \arrow[r, "a_{A,B,C}"] \arrow[d, "(D_A+D_B)+D_C"'] & A\times(B\times C) \arrow[d, "D_A+(D_B+D_C)"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) =$$

$$\begin{array}{rccl} \Psi(\mathrm{id}, a_{A,B,C})\colon (\Psi(A)\times\Psi(B))\times\Psi(C) & \longrightarrow & \Psi(A)\times(\Psi(B)\times\Psi(C)) \\ ((a,b),c) & \longmapsto & \{(a,(b,c))\} \end{array} = a_{\Psi(A),\Psi(B),\Psi(C)}.$$

$\blacksquare$

Chapter 4

Decomposition Spaces

In this last chapter, we consider a discrete variant of decomposition spaces, which define by construction comonoid objects in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$ and, adding a monoidal structure to such decomposition spaces, we also induce bimonoid objects automatically. Finally, we give a detailed example that shows how this construction works. Such constructions are similarly done with ∞-groupoids in, for instance, [GCKT18d].

4.1 Simplicial Objects

In order to be able to talk about decomposition spaces, we first need to know the basics of simplicial objects. The main reference that we use is [Wei95].

Definition 4.1.1. The *simplex category* Δ is the category consisting of non-empty finite ordinals

$$[n] = \{0 < 1 < \cdots < n\}\,,\, n \geq 0,$$

as objects and order-preserving maps as morphisms.

If $\mathcal{C}$ is any category, a *simplicial object* in $\mathcal{C}$ a $\mathcal{C}$-valued presheaf on Δ, that is, a contravariant functor

$$\begin{array}{rcl} X\colon \Delta^{op} & \longrightarrow & \mathcal{C} \\ [n] & \longmapsto & X_n := X([n]), \\ [n] \longrightarrow [m] & \longmapsto & X_m \longrightarrow X_n, \end{array}$$

and the category of *simplicial objects* in $\mathcal{C}$ is defined as the functor category

$$\mathbf{Funct}(\Delta^{op}, \mathcal{C}).$$

Dually, the category of *cosimplicial objects* in $\mathcal{C}$ is defined as

$$\mathbf{Funct}(\Delta, \mathcal{C}).$$

In order to make sense of these categorical constructions, it is useful to study these categories in a more combinatorial way. Let us look at the morphisms in the simplex category: given $m, n \in \mathbb{N}$, it is clear that giving an order-preserving map $[n] \longrightarrow [m]$ in Δ is the same as giving a sequence of integer numbers

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_n \leq m,$$

so the number of order-preserving maps between the ordinals $[n]$ and $[m]$ is given by the binomial coefficient $\binom{n+m+1}{m}$. Therefore, it may be useful to introduce some distinguished maps to work with, instead of trying to deal with the whole set of morphisms. The prominent maps are the so-called *coface maps* and *codegeneracy maps*.

Definition 4.1.2. For every integer $n \geq 1$ and every $k \in \{0, \ldots, n\}$, the *coface map*

$$d_n^k \colon [n-1] \longrightarrow [n]$$

is the unique injective map in Δ such that its image misses k. That is,

$$d_n^k(i) = \begin{cases} i, & \text{if } i < k, \\ i+1, & \text{if } i \geq k. \end{cases}$$

Moreover, for every $n \geq 0$ and every $k \in \{0, \ldots, n\}$, the *codegeneracy map*

$$s_n^k \colon [n+1] \longrightarrow [n]$$

is the unique surjective map in Δ which hits k twice. The map is then given by

$$s_n^k(i) = \begin{cases} i, & \text{if } i \leq k, \\ i-1, & \text{if } i > k. \end{cases}$$

We usually drop the subscripts and write just d^k and s^k. The maps with $0 < k < n$ are termed *inner*, and *outer* otherwise. We write $d^{\perp} := d^0$ (*bottom coface map*) and $d^{\top} := d^n$ (*top coface map*) for the outer coface maps.

If $X \in \mathbf{Funct}(\Delta^{op}, \mathcal{C})$ is a simplicial object in $\mathcal{C}$ we write, for every $n \geq 0$ and every $k \in \{0, \ldots, n\}$, $X(d^k) = d_k$ and $X(s^k) = s_k$, and we call them face and degeneracy maps, respectively.

Coface and codegeneracy maps verify some important relations between them, which are called the *cosimplicial identities.*

Proposition 4.1.3. *The following relations in Δ hold:*

1. $d^j \circ d^i = d^i \circ d^{j-1}$, *if* $i < j$,
2. $s^j \circ s^i = s^i \circ s^{j+1}$, *if* $i \leq j$,
3. $s^j \circ d^i = \begin{cases} d^i \circ s^{j-1}, & \text{if } i < j, \\ \mathrm{id}, & \text{if } i = j \text{ or } i = j+1, \\ d^{i-1} \circ s^j, & \text{if } i > j+1. \end{cases}$

The importance of these maps lies in the property that every morphism in Δ can be written as a composition of coface and codegeneracy maps in a certain unique way (see [**?**], 8.1). From this fact it follows that Δ can be identified with the free category generated by the ordinals $[n]$, the cofaces and the codegeneracies modulo the relations 4.1.3. This fact ensures that we only need to give a particular data to define a simplicial object, which is useful both in theory and practice.

Proposition 4.1.4. *Given a category $\mathcal{C}$, there is a one-to-one correspondence between simplicial objects in $\mathcal{C}$ and the following data: a sequence of objects $\{X_n\}_{n\geq 0}$ in $\mathcal{C}$ together with two families of morphisms*

$$\{d_k \colon X_n \longrightarrow X_{n-1} : n \geq 1,\, 0 \leq k \leq n\} \text{ and } \{s_k \colon X_n \longrightarrow X_{n+1} : n \geq 0,\, 0 \leq k \leq n\}$$

also called faces and degeneracies, respectively, such that the dualized identities in 4.1.3, named simplicial identities, are satisfied:

1. $d_i \circ d_j = d_{j-1} \circ d_i$, *if* $i < j$,
2. $s_i \circ s_j = s_{j+1} \circ s_i$, *if* $i \leq j$,
3. $d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i, & \text{if } i < j, \\ \mathrm{id}, & \text{if } i = j \text{ or } i = j+1, \\ s_j \circ d_{i-1}, & \text{if } i > j+1. \end{cases}$

Under this correspondence, if $X \in \mathbf{Funct}(\Delta^{op}, \mathscr{C})$ is a simplicial object in $\mathcal{C}$, $d_i = X(d^i)$ and $s_i = X(s^i)$.

Dualizing the previous proposition, we get an analogous result for cosimplicial objects. Let us see now a classical example, whose geometric interpretation is the cause of the origin of the terms face and degeneracy maps.

Example 4.1.5 (Simplices)**.** *Recall that, given an integer $n \geq 0$, the geometric n-simplex Δ^n is the subspace of $\mathbb{R}^{n+1}$ defined by*

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \in [0,1]^n : \sum_{i=0}^{n+1} x_i = 1 \right\}.$$

*Let $\{e_i\}_{0\leq i\leq n}$ be the standard basis of $\mathbb{R}^{n+1}$, that is, the vertices of Δ^n. Identifying each $i \in [n]$ with the corresponding e_i, we can think of a map $\alpha\colon [n] \longrightarrow [m]$ as a map from the vertices of Δ^n to the vertices of Δ^m by setting $\alpha(e_i) := e_{\alpha(i)}$. Every map $\alpha\colon [n] \longrightarrow [m]$ between the vertices of Δ^n and Δ^m can be linearly extended to a continuous map $\alpha_*\colon \Delta^n \longrightarrow \Delta^m$, so that this construction gives us a cosimplicial topological space $\Delta \longrightarrow$ **Top**. Geometrically, the coface map d^k induces the inclusion of Δ^{n-1} into Δ^n as its k-th face, that is, the face opposite to the k-th vertex of Δ^n, and the codegeneracy map induces the projection $\Delta^{n+1} \longrightarrow \Delta^n$ onto the k-th face, which identifies the vertices k and $k+1$. As we have said, this example clarifies why these maps are called as they are.*

4.2 Discrete Decomposition Spaces

Since cell-sets are nothing but discrete groupoids over $\mathbb{Z}$, we are forced to establish a discrete variant of the general concept of decomposition space.

Definition 4.2.1. Let $f\colon [n] \longrightarrow [m]$ be a morphism in Δ. We say that f is *active* (also called *generic*) if it preserves end points, that is, if $f(0) = 0$ and $f(n) = m$. We also say that f is *inert* (also called *free*) if it is distance preserving, which means that $f(i+1) = f(i) + 1$ for every $0 \leq i \leq n-1$.

Definition 4.2.2. Let $X\colon \Delta^{op} \longrightarrow \mathbf{DiscGrpd}/\mathbb{Z}$ be a simplicial object in $\mathbf{DiscGrpd}/\mathbb{Z}$. We say that X is a *graded discrete decomposition space* if the following conditions hold:

1. X takes generic-free pushouts to pullbacks, that is, for every pushout square in Δ

$$\begin{tikzcd} {[p]} & {[m]} \arrow[l, "g'"'] \\ {[q]} \arrow[u, "f'"] \arrow[ur, phantom, "\lrcorner"] & {[n]} \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

   such that g is generic and f is free, its image under X

$$X\left(\begin{tikzcd} {[p]} & {[m]} \arrow[l, "g'"'] \\ {[q]} \arrow[u, "f'"] \arrow[ur, phantom, "\lrcorner"] & {[n]} \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}\right) = \begin{tikzcd}[row sep=small, column sep=small] & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\ X_p \arrow[ur] \arrow[rr] \arrow[dd] & & X_m \arrow[ur] \arrow[dd] & \\ & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\ X_q \arrow[ur] \arrow[rr] & & X_n \arrow[ur] & \end{tikzcd}$$

   is a pullback in $\mathbf{DiscGrpd}/\mathbb{Z}$.

2. *Dimension property.* For every n, the dimension functors $D_n\colon X_n \longrightarrow \mathbb{Z}$ verify the following conditions:

   (a) $D_0(x) = 0$, for every $x \in X_0$,

   (b) $(D_{n+1} \circ s_j)(x) = D_n(x)$, for every $0 \leq j \leq n$ and for every $x \in X_n$,

   (c) $(D_{n-1} \circ d_i)(x) = D_n(x)$, for every $0 < i < n$ and for every $x \in X_n$,

   (d) $(D_{n-1} \circ d_0)(x) = D_n(x) - (D_1 \circ d_\top^{n-1})(x)$, for every $x \in X_n$,

   (e) $(D_{n-1} \circ d_n)(x) = D_n(x) - (D_1 \circ d_\perp^{n-1})(x)$, for every $x \in X_n$.

In fact, there is a simpler class of pullback squares that one has to check, as the following proposition claims. Its proof can be found, stated in terms of general decomposition spaces, in [GCKT18d], proposition 3.5.

Proposition 4.2.3. *Let $X\colon \Delta^{op} \longrightarrow \mathbf{DiscGrpd}/\mathbb{Z}$ be a simplicial object in $\mathbf{DiscGrpd}/\mathbb{Z}$. Then, X takes generic-free pushouts to pullbacks if and only if the following commutative cubes are pullbacks*

$$
\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] && \mathbb{Z} \arrow[dd, equal] \\
X_1 \arrow[ur] \arrow[rr, "s_1"] \arrow[dd, "d_\perp"'] && X_2 \arrow[ur] \arrow[dd, "d_\perp" near start] & \\
& \mathbb{Z} \arrow[rr, equal] && \mathbb{Z} \\
X_0 \arrow[ur] \arrow[rr, "s_0"] && X_1 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] && \mathbb{Z} \arrow[dd, equal] \\
X_1 \arrow[ur] \arrow[rr, "s_0"] \arrow[dd, "d_\top"'] && X_2 \arrow[ur] \arrow[dd, "d_\top" near start] & \\
& \mathbb{Z} \arrow[rr, equal] && \mathbb{Z} \\
X_0 \arrow[ur] \arrow[rr, "s_0"] && X_1, \arrow[ur] &
\end{tikzcd}
$$

and the following commutative cubes are pullbacks for every $n \geq 2$ and some $0 < i = i(n) < n$, depending on n

$$
\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] && \mathbb{Z} \arrow[dd, equal] \\
X_{n+1} \arrow[ur] \arrow[rr, "d_{i+1}"] \arrow[dd, "d_\perp"'] && X_n \arrow[ur] \arrow[dd, "d_\perp" near start] & \\
& \mathbb{Z} \arrow[rr, equal] && \mathbb{Z} \\
X_n \arrow[ur] \arrow[rr, "d_i"] && X_{n-1} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] && \mathbb{Z} \arrow[dd, equal] \\
X_{n+1} \arrow[ur] \arrow[rr, "d_i"] \arrow[dd, "d_\top"'] && X_n \arrow[ur] \arrow[dd, "d_\top" near start] & \\
& \mathbb{Z} \arrow[rr, equal] && \mathbb{Z} \\
X_n \arrow[ur] \arrow[rr, "d_i"] && X_{n-1}. \arrow[ur] &
\end{tikzcd}
$$

Given two simplicial objects X and Y, a simplicial map $F\colon X \longrightarrow Y$ is just a natural transformation. In the case of decomposition spaces we need to restrict ourselves to a special subclass of simplicial maps, called CULF maps, whose definition is intended to induce coalgebra morphisms in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$.

Definition 4.2.4. Let $F\colon X \longrightarrow Y$ be a simplicial map between simplicial objects in $\mathbf{DiscGrpd}/\mathbb{Z}$. We say that F is *ULF (unique lifting of factorizations)* if it is cartesian on each inner coface map in Δ, that is, if for every $n \geq 1$, the naturality cubes

$$
\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] && \mathbb{Z} \arrow[dd, equal] \\
X_n \arrow[ur] \arrow[rr, "d_i"] \arrow[dd, "F_n"'] \arrow[dr, phantom, "\lrcorner", very near start] && X_{n-1} \arrow[ur] \arrow[dd, "F_{n-1}" near start] & \\
& \mathbb{Z} \arrow[rr, equal] && \mathbb{Z} \\
Y_n \arrow[ur] \arrow[rr, "d_i"] && Y_{n-1} \arrow[ur] &
\end{tikzcd}
$$

are pullbacks for every $0 < i < n$. It is called *conservative* if it is cartesian on each codegeneracy map in Δ, which means that for every $n \geq 0$, the naturality cubes

$$
\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_n \arrow[ur] \arrow[rr, "s_i"] \arrow[dd, "F_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & & X_{n+1} \arrow[ur] \arrow[dd, "F_{n+1}", near start] & \\
& \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
Y_n \arrow[ur] \arrow[rr, "s_i"] & & Y_{n+1} \arrow[ur] &
\end{tikzcd}
$$

are pullbacks for every $0 \leq i \leq n$. If both conditions hold, we say that F is a *CULF* map.

An important example of CULF maps are the natural isomorphisms in the simplicial category $\mathbf{Funct}(\Delta^{op}, \mathbf{DiscGrpd}/\mathbb{Z})$. This fact is a direct consequence of the following simple lemma.

Lemma 4.2.5. *Let $\mathcal{C}$ be a category with pullbacks and consider a commutative square in $\mathcal{C}$*

$$
\begin{tikzcd}
A \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}
$$

such that g and $\widetilde{g}$ are isomorphisms. Then, such a square is a pullback.

Proof. The proof is straightforward. Given any span

$$C \xleftarrow{h} E \xrightarrow{i} B$$

in $\mathcal{C}$ such that the outer square within the diagram

$$
\begin{tikzcd}
E \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "h"'] & & \\
& A \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] & B \arrow[d, "g"] \\
& C \arrow[r, "f"'] & D
\end{tikzcd}
$$

commutes, we define $\varphi := \widetilde{g}^{-1} \circ h$, which makes sense because $\widetilde{g}$ is an isomorphism. One can easily check that it is the unique morphism such that the pullback definition is satisfied. ■

Now we have all the ingredients to define our category of discrete decomposition spaces.

Definition 4.2.6. We denote by **GrDecomp** the subcategory of

$$\mathbf{Funct}(\Delta^{op}, \mathbf{DiscGrpd}/\mathbb{Z})$$

consisting of graded discrete decomposition spaces and CULF maps with the additional property that, for every $X \in \mathbf{GrDecomp}$, the spans

$$\begin{tikzcd} X_1 \arrow[d, "D_1"'] & X_2 \arrow[l, "d_1"'] \arrow[r, "(d_2,d_0)"] \arrow[d, "D_2"] & X_1 \times X_1 \arrow[d, "D_1+D_1"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_1 \arrow[d, "D_1"'] & X_0 \arrow[l, "s_0"'] \arrow[r, "t"] \arrow[d, "0"] & 1 \arrow[d, "0"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

have the p-property and are finite.

4.3 Comonoid Objects in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$

We are now interested in a functor

$$\Phi\colon \mathbf{GrDecomp} \longrightarrow \mathbf{Comon}(\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})),$$

whose existence will be guaranteed precisely because of the decomposition space axioms. In objects, we define it as

$$\Phi(X) := (X_1 \xrightarrow{D_1} \mathbb{Z}, \delta_X, \varepsilon_X),$$

where

$$\delta_X = \begin{tikzcd} X_1 \arrow[d, "D_1"'] & X_2 \arrow[l, "d_1"'] \arrow[r, "(d_2,d_0)"] \arrow[d, "D_2"] & X_1 \times X_1 \arrow[d, "2D_1"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

and

$$\varepsilon_X = \begin{tikzcd} X_1 \arrow[d, "D_1"'] & X_0 \arrow[l, "s_0"'] \arrow[r, "t"] \arrow[d, "0"] & 1 \arrow[d, "0"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Given a CULF map $F\colon X \longrightarrow Y$, we just regard the component F_1 as a span in $\mathbf{DiscGrpd}/\mathbb{Z}$ adding identities

$$\Phi(F) := \begin{tikzcd} X_1 \arrow[d, "D_1^X"'] & X_1 \arrow[l, equal] \arrow[r, "F_1"] \arrow[d, "D_1^X"] & Y_1 \arrow[d, "D_1^Y"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Lemma 4.3.1. *Let $\mathcal{C}$ be any category and consider the following commutative diagram in $\mathcal{C}$*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{(I)}"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{(II)}"] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F.
\end{tikzcd}$$

Suppose that the right-hand inner square (II) *is a pullback. Then,*

$$\textit{the left-hand inner square } \mathrm{(I)} \textit{ is a pullback} \iff \textit{the outer square is.}$$

Proposition 4.3.2. *The assignment Φ is well-defined and it is a functor.*

Proof. Let $X \in \mathbf{GrDecomp}$ be a graded discrete decomposition space. We will first show that the span δ_X is coassociative, that is, we will show that the diagram

$$\begin{tikzcd}
X_1 \arrow[rr, "\delta_X"] \arrow[d, "\delta_X"'] & & X_1 \otimes X_1 \arrow[d, "\delta_X \otimes \mathrm{id}_{X_1}"] \\
X_1 \otimes X_1 \arrow[r, "\mathrm{id}_{X_1} \otimes \delta_X"'] & X_1 \otimes X_1 \otimes X_1 &
\end{tikzcd}$$

is commutative in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$ or, what is the same, we will see that there is an isomorphism of spans between $(\delta_X \otimes \mathrm{id}_{X_1}) \circ \delta_X$ and $(\mathrm{id}_{X_1} \otimes \delta_X) \circ \delta_X$. If we expand the above diagram,

$$\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_1 \arrow[ur, "D_1"] & & X_2 \arrow[ll, "d_1"'] \arrow[ur, "D_2"] \arrow[rr, "{(d_2,d_0)}"] & & X_1 \times X_1 \arrow[ur, "2D_1"] & \\
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_2 \arrow[uu, "d_1"] \arrow[ur, "D_2"] \arrow[dd, "{(d_2,d_0)}"'] & & X_3 \arrow[ll, "d_2"'] \arrow[uu, "d_1"'] \arrow[ur, "D_3"] \arrow[rr, "{(d_3,\, d_0 \circ d_0)}"] \arrow[dd, "{(d_2 \circ d_2,\, d_0)}"] & & X_2 \times X_1 \arrow[uu, "d_1 \times \mathrm{id}_{X_1}"'] \arrow[ur, "D_2+D_1"'] \arrow[dd, "{(d_2,\, d_0) \times \mathrm{id}_{X_1}}"] & \\
& \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_1 \times X_1 \arrow[ur, "2D_1"] & & X_1 \times X_2 \arrow[ll, "\mathrm{id}_{X_1} \times d_1"'] \arrow[ur, "D_1+D_2"] \arrow[rr, "{\mathrm{id}_{X_1} \times (d_2,d_0)}"] & & X_1 \times X_1 \times X_1 \arrow[ur, "3D_1"'] &
\end{tikzcd}$$

we observe that there exists such an isomorphism of spans if both the top right-hand cube and the bottom left-hand cube are pullbacks as indicated in the diagram. Let us focus first on the top right-hand cube. Using lemma 4.3.1, we know that it will be a pullback if and only if the outer cube of the following diagram is

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_2 \arrow[ur, "D_2"] \arrow[rr, "{(d_2,d_0)}"] & & X_1 \times X_1 \arrow[ur, "2D_1"] \arrow[rr, "\mathrm{pr}_1"] & & X_1 \arrow[ur, "D_1"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_3 \arrow[uu, "d_1"] \arrow[ur, "D_3"] \arrow[rr, "{(d_3,d_0\circ d_0)}"] & & X_2 \times X_1 \arrow[uu, "d_1\times\mathrm{id}_{X_1}"] \arrow[ur, "D_2+D_1"] \arrow[rr, "\mathrm{pr}_1"] & & X_2, \arrow[uu, "d_1"] \arrow[ur, "D_2"] &
\end{tikzcd}
$$

but the outer cube is a pullback as a consequence of one of the decomposition space axioms. The argument for the bottom left-hand cube is completely analogous.

Now let us check that ε_X is a counit, that is, we need to see if the following diagram

$$
\begin{tikzcd}
 & X_1 \otimes 1 & \\
X_1 \arrow[ur, "\rho_{X_1}"] \arrow[rr, "\delta_X"] \arrow[dr, "\lambda_{X_1}"'] & & X_1 \otimes X_1 \arrow[ul, "\mathrm{id}_{X_1}\otimes\varepsilon_X"'] \arrow[dl, "\varepsilon_X\otimes\mathrm{id}_{X_1}"] \\
 & 1 \otimes X_1 &
\end{tikzcd}
$$

is commutative up to isomorphism of spans. Expanding the lower triangle,

$$
\begin{tikzcd}
X_1 \arrow[d] \arrow[ddr, equal] & & X_2 \arrow[ll, "d_1"'] \arrow[rr, "{(d_2,d_0)}"] \arrow[d] & & X_1 \times X_1 \arrow[d] \\
\mathbb{Z} \arrow[rr, equal] \arrow[ddr, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[ddl, equal] & & \mathbb{Z} \arrow[ddl, equal] \\
 & X_1 \arrow[uur, "s_1"] \arrow[rr, "{(d_0,\mathrm{id}_{X_1})}"] \arrow[d] \arrow[ddr, "\lambda_{X_1}"] & & X_0 \times X_1 \arrow[uur, "s_0\times\mathrm{id}_{X_1}"] \arrow[d] \arrow[ddl, "t\times\mathrm{id}_{X_1}"'] & \\
 & \mathbb{Z} \arrow[rr, equal] \arrow[ddr, equal] & & \mathbb{Z} \arrow[ddl, equal] & \\
 & & 1 \times X_1 \arrow[d] & & \\
 & & \mathbb{Z} & &
\end{tikzcd}
$$

we see that it commutes up to isomorphism if the parallelogram prism is a pullback. Again, using lemma 4.3.1, we can deduce that it will be a pullback if and only if the outer prism in the diagram

$$\begin{tikzcd}
 & X_2 \arrow[rr, "{(d_2,d_0)}"] \arrow[d] & & X_1 \times X_1 \arrow[rr, "\mathrm{pr}_2"] \arrow[d] & & X_1 \arrow[d] \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_1 \arrow[uur, "s_1"] \arrow[rr, "{(d_0,\mathrm{id}_{X_1})}"] \arrow[d] & & X_0 \times X_1 \arrow[uur, "s_0 \times \mathrm{id}_{X_1}"] \arrow[rr, "\mathrm{pr}_1"] \arrow[d] & & X_0 \arrow[uur, "s_0"] \arrow[d] & \\
\mathbb{Z} \arrow[rr, equal] \arrow[uur, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[uur, equal] & & \mathbb{Z} \arrow[uur, equal] &
\end{tikzcd}$$

is a pullback. But, once again, this is true because of the decomposition space axioms.

Now given a CULF map

$$F \colon X \longrightarrow Y,$$

we want to see that the span

$$\Phi(F) = \begin{tikzcd}
X_1 \arrow[r, equal] \arrow[d, "D_1^X"'] & X_1 \arrow[r, "F_1"] \arrow[d, "D_1^X"] & Y_1 \arrow[d, "D_1^Y"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}$$

is a coalgebra morphism. That is, we want to verify that the following diagram commutes up to isomorphism of spans

$$\begin{tikzcd}
X_1 \arrow[r, "\Phi(F)"] \arrow[d, "\delta_X"'] & Y_1 \arrow[d, "\delta_Y"] \\
X_1 \otimes X_1 \arrow[r, "\Phi(F) \otimes \Phi(F)"'] & Y_1 \otimes Y_1.
\end{tikzcd}$$

If we expand this diagram,

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_1 \arrow[ur, "D_1^X"] \arrow[rr, equal] & & X_1 \arrow[ur, "D_1^X"] \arrow[rr, "F_1"] & & Y_1 \arrow[ur, "D_1^Y"] & \\
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_2 \arrow[uu, "d_1^X"] \arrow[dd, "{(d_2^X, d_0^X)}"'] \arrow[ur, "D_2^X"] \arrow[rr, equal] & & X_2 \arrow[uu, "d_1^X"] \arrow[dd, "{(d_2^X,\, d_0^X)}"] \arrow[ur, "D_2^X"] \arrow[rr, "F_2"] & & Y_2 \arrow[uu, "d_1^Y"] \arrow[dd, "{(d_2^Y,\, d_0^Y)}"] \arrow[ur, "D_2^Y"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_1 \times X_1 \arrow[ur, "2D_1^X"] \arrow[rr, equal] & & X_1 \times X_1 \arrow[ur, "2D_1^X"] \arrow[rr, "F_1 \times F_1"] & & Y_1 \times Y_1 \arrow[ur, "2D_1^Y"] &
\end{tikzcd}
$$

we see that what we want to achieve is just a consequence of being CULF. Indeed, the upper right-hand cube is a pullback by definition of CULF map, and the lower left-hand cube is trivially a pullback.

The last step is to see that this construction is functorial. For every graded discrete decomposition space X,

$$
\Phi(\mathrm{id}_X \colon X \longrightarrow X) =
\begin{tikzcd}
X_1 \arrow[r, equal] \arrow[d, "D_1"'] & X_1 \arrow[r, equal] \arrow[d, "D_1"] & X_1 \arrow[d, "D_1"] \\
\mathbb{Z} \arrow[r, equal] & X_1 \arrow[r, equal] & \mathbb{Z}.
\end{tikzcd}
= \mathrm{id}_{\Phi(X_1)}\,.
$$

Now given a pair of CULF maps

$$X \xrightarrow{F} Y \xrightarrow{G} Z,$$

we want to see if there exists an isomorphism of spans

$$\Phi(G \circ F) \cong \Phi(G) \circ \Phi(F).$$

This is easy to verify, since

$$\Phi(G) \circ \Phi(F) =$$

$$
\begin{tikzcd}
 & & X_1 \arrow[dl, equal] \arrow[d] \arrow[dr, "F_1"] & & \\
X_1 \arrow[r, equal] \arrow[d] & X_1 \arrow[d] \arrow[dr, "F_1"] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & Y_1 \arrow[r, "G_1"] \arrow[d] \arrow[dl, equal] & Z_1 \arrow[d] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[dr, equal] & Y_1 \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[dl, equal] & \mathbb{Z}. \\
 & & \mathbb{Z} & &
\end{tikzcd}
=
$$

$$\begin{tikzcd} X_1 \arrow[r, equal] \arrow[d, "D_1^X"'] & X_1 \arrow[r, "G_1 \circ F_1"] \arrow[d, "D_1^X"] & Z_1 \arrow[d, "D_1^Z"] \\ \mathbb{Z} \arrow[r, equal] & X_1 \arrow[r, equal] & \mathbb{Z}. \end{tikzcd} = \Phi(G \circ F).$$

Thus, Φ is a well-defined functor. ■

4.4 Extension to Bimonoid Objects

Now we address the problem of equipping **GrDecomp** with a symmetric monoidal structure. First, we need to equip **DiscGrpd**/$\mathbb{Z}$ with such an structure.

Proposition 4.4.1. *Consider the assignment*

$$\otimes \colon \mathbf{DiscGrpd}/\mathbb{Z} \times \mathbf{DiscGrpd}/\mathbb{Z} \longrightarrow \mathbf{DiscGrpd}/\mathbb{Z}$$

defined as follows: in objects, we have that

$$\left(\begin{tikzcd} A \arrow[d, "D_A"] \\ \mathbb{Z} \end{tikzcd}, \begin{tikzcd} B \arrow[d, "D_B"] \\ \mathbb{Z} \end{tikzcd} \right) \longmapsto \left(\begin{tikzcd} A \arrow[d, "D_A"] \\ \mathbb{Z} \end{tikzcd} \right) \otimes \left(\begin{tikzcd} B \arrow[d, "D_B"] \\ \mathbb{Z} \end{tikzcd} \right) := \begin{tikzcd} A \times B \arrow[d, "D_A \times D_B"] \\ \mathbb{Z} \times \mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z}, \end{tikzcd}$$

and in arrows it acts as

$$\left(\begin{tikzcd} A_1 \arrow[rr, "f"] \arrow[dr, "D_{A_1}"'] & & B_1 \arrow[dl, "D_{B_1}"] \\ & \mathbb{Z} & \end{tikzcd} \right) \otimes \left(\begin{tikzcd} A_2 \arrow[rr, "g"] \arrow[dr, "D_{A_2}"'] & & B_2 \arrow[dl, "D_{B_2}"] \\ & \mathbb{Z} & \end{tikzcd} \right) := \begin{tikzcd} A_1 \times A_2 \arrow[r, "f \times g"] \arrow[d, "D_{A_1} \times D_{A_2}"'] & B_1 \times B_2 \arrow[d, "D_{B_1} \times D_{B_2}"] \\ \mathbb{Z} \times \mathbb{Z} \arrow[d, "+"'] & \mathbb{Z} \times \mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Then, $\otimes$ *is a functor and it equips* **DiscGrpd**/$\mathbb{Z}$ *with a symmetric monoidal structure.*

Proof. For simplicity, let us write $F := \otimes$, so that

$$F(A, B) := A \otimes B \text{ and } F(f, g) := f \otimes g,$$

omitting the dimension functors. For every pair of objects $(A, B) \in \mathbf{DiscGrpd}/\mathbb{Z} \times \mathbf{DiscGrpd}/\mathbb{Z}$,

$$F(\mathrm{id}_{(A,B)}) = F(\mathrm{id}_A, \mathrm{id}_B) = \begin{tikzcd} A \times B \arrow[r, "\mathrm{id}_A \times \mathrm{id}_B"] \arrow[d, "D_A \times D_B"'] & A \times B \arrow[d, "D_A \times D_B"] \\ \mathbb{Z} \times \mathbb{Z} \arrow[d, "+"'] & \mathbb{Z} \times \mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd} =$$

$$\begin{tikzcd} A\times B \arrow[r, equal, "\mathrm{id}_{A\times B}"] \arrow[d, "D_A\times D_B"'] & A\times B \arrow[d, "D_A\times D_B"] \\ \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"'] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd} = \mathrm{id}_{F(A,B)},$$

since $1_A \times 1_B = 1_{A\times B}$ just by functoriality of the cartesian product. For the second property, consider any sequence of composable arrows

$$\left(\begin{array}{cc} A_1 & A_2 \\ \downarrow D_{A_1}, & \downarrow D_{A_2} \\ \mathbb{Z} & \mathbb{Z} \end{array}\right) \xrightarrow{(f,g)} \left(\begin{array}{cc} B_1 & B_2 \\ \downarrow D_{B_1}, & \downarrow D_{B_2} \\ \mathbb{Z} & \mathbb{Z} \end{array}\right) \xrightarrow{(h,i)} \left(\begin{array}{cc} C_1 & C_2 \\ \downarrow D_{C_1}, & \downarrow D_{C_2} \\ \mathbb{Z} & \mathbb{Z} \end{array}\right).$$

On the one hand,

$$F((h,i)\circ(f,g)) = F(h\circ f, i\circ g) = \begin{tikzcd} A_1\times A_2 \arrow[r, "(h\circ f)\times(i\circ g)"] \arrow[d, "D_{A_1}\times D_{A_2}"'] & C_1\times C_2 \arrow[d, "D_{C_1}\times D_{C_2}"] \\ \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"'] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

On the other hand,

$$F(h,i)\circ F(f,g) =$$

$$\left(\begin{tikzcd} B_1\times B_2 \arrow[r, "h\times i"] \arrow[d, "D_{B_1}\times D_{B_2}"'] & C_1\times C_2 \arrow[d, "D_{C_1}\times D_{C_2}"] \\ \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"'] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) \circ \left(\begin{tikzcd} A_1\times A_2 \arrow[r, "f\times g"] \arrow[d, "D_{A_1}\times D_{A_2}"'] & B_1\times B_2 \arrow[d, "D_{B_1}\times D_{B_2}"] \\ \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"'] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}\right) =$$

$$= \begin{tikzcd} A_1\times A_2 \arrow[r, "(h\times i)\circ(f\times g)"] \arrow[d, "D_{A_1}\times D_{A_2}"'] & C_1\times C_2 \arrow[d, "D_{C_1}\times D_{C_2}"] \\ \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"'] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Hence, it only remains to check that

$$(h \circ f) \times (i \circ g) = (h \times i) \circ (f \times g),$$

but this is trivially true thanks to, again, the functoriality of the cartesian product. Therefore, $\otimes$ is a functor. Now it remains to determine the unit, the associator, the left and right unitors and see that the compatibility conditions hold. At this point, the reader may have observed that the proposition follows essentially because **DiscGrpd** together with the categorical product has a cartesian symmetric monoidal structure, but anyway we give some of the details. Our candidate to be the unit is the terminal groupoid together with the zero dimension functor

$$I = \begin{array}{c} 1 \\ \big\downarrow{\scriptstyle 0} \\ \mathbb{Z}, \end{array}$$

as we will see later. Given now three discrete groupoids over $\mathbb{Z}$,

$$\begin{array}{ccc} A & B & C \\ \big\downarrow{\scriptstyle D_A} & \big\downarrow{\scriptstyle D_B} & \big\downarrow{\scriptstyle D_C} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z}, \end{array}$$

let

$$\begin{aligned} a = a_{A,B,C} \colon (A \times B) \times C &\longrightarrow A \times (B \times C) \\ ((x, y), z) &\longmapsto (x, (y, z)) \end{aligned}$$

be the associator in **DiscGrpd**. Therefore, our associator in **DiscGrpd**/$\mathbb{Z}$ must be

$$a = a_{A,B,C} = \begin{tikzcd} (A \times B) \times C \arrow[r, "a_{A,B,C}"] \arrow[d, "(D_A+D_B)\times D_C"'] & A \times (B \times C) \arrow[d, "D_A\times(D_B+D_C)"] \\ \mathbb{Z} \times \mathbb{Z} \arrow[d, "+"'] & \mathbb{Z} \times \mathbb{Z} \arrow[d, "+"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Indeed, the above diagram obviously commutes, and the naturality trivially follows from the naturality of the associator a in **DiscGrpd**, since composition in the slice category is the same as composition in the original category. The same happens with the unitors: let

$$\begin{aligned} \lambda = \lambda_A \colon 1 \times A &\longrightarrow A \\ (*, a) &\longmapsto a \end{aligned}$$

and

$$\begin{aligned} \rho = \rho_A \colon A \times 1 &\longrightarrow A \\ (a, *) &\longmapsto a \end{aligned}$$

be the left and right unitors in **DiscGrpd**, respectively. Then, the natural extensions to **DiscGrpd**/$\mathbb{Z}$

$$\lambda = \lambda_A = \begin{tikzcd} 1 \times A \arrow[r, "\lambda_A"] \arrow[d, "0+D_A"] & A \arrow[d, "D_A"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd} \qquad \rho = \rho_A = \begin{tikzcd} A \times 1 \arrow[r, "\rho_A"] \arrow[d, "D_A+0"] & A \arrow[d, "D_A"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

are obviously the left and right unitors in **DiscGrpd**/$\mathbb{Z}$. The braiding is defined in a similar way and compatibility conditions trivially hold, since they hold in **DiscGrpd**. ■

The following proposition shows that the structure we just have defined induces a symmetric monoidal structure on the category of simplicial objects in **DiscGrpd**/$\mathbb{Z}$.

Proposition 4.4.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and assume that $(\mathcal{D}, \otimes, I, B)$ has a symmetric monoidal structure. Then, the functor category* $\mathbf{Funct}(\mathcal{C}, \mathcal{D})$ *has a symmetric monoidal structure as well, given pointwise.*

Proof. We define a bifunctor

$$\Box\colon \mathbf{Funct}(\mathcal{C}, \mathcal{D}) \times \mathbf{Funct}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathbf{Funct}(\mathcal{C}, \mathcal{D})$$

by setting

$$(F \Box G)(c) := F(c) \otimes G(c) \qquad (F \Box G)(f) := F(f) \otimes G(f),$$

for every $F, G \in \mathbf{Funct}(\mathcal{C}, \mathcal{D})$, for every $c \in \mathcal{C}$ and for every $f \in \mathrm{Hom}(\mathcal{C})$. In natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "G"{name=U}] \arrow[r, bend right=50, "G'"'{name=D}] & \mathcal{D}, \arrow[from=U, to=D, Rightarrow, "\beta"] \end{tikzcd}$$

we define

$$(\alpha \Box \beta)_c := F(c) \otimes G(c) \xrightarrow{\alpha_c \otimes \beta_c} F'(c) \otimes G'(c),$$

for every $\alpha \in \mathbf{Nat}(F, F')$, $\beta \in \mathbf{Nat}(G, G')$, for every $F, F', G, G' \in \mathbf{Funct}(\mathcal{C}, \mathcal{D})$ and for every $c \in \mathcal{C}$. The unit $\widetilde{I}$ is given by the constant functor at the unit of $\mathcal{D}$

$$\begin{array}{rcl} \widetilde{I}\colon \mathcal{C} & \longrightarrow & \mathcal{D} \\ c & \longmapsto & I \\ c \longrightarrow c' & \longmapsto & \mathrm{id}_I, \end{array}$$

and the braiding $\widetilde{B}$ is given, for every $F, G \in \mathbf{Funct}(\mathcal{C}, \mathcal{D})$, by the natural isomorphism

$$\widetilde{B}_{F,G}\colon F \Box G \longrightarrow G \Box F,$$

where each component is also a natural isomorphism, given by

$$(\widetilde{B}_{F,G})(c) = F(c) \otimes G(c) \xrightarrow{B_{F(c),G(c)}} G(c) \otimes F(c).$$

The rest of the structure is constructed in a similar way, and using the fact that the required axioms hold pointwise, it is easy to see that they also hold in the functor category. ∎

Applying such proposition to the category of simplicial objects in $\mathbf{DiscGrpd}/\mathbb{Z}$, we have that $\mathbf{Funct}(\Delta^{op}, \mathbf{DiscGrpd}/\mathbb{Z})$ inherits a symmetric monoidal structure. The problem of equipping $\mathbf{GrDecomp}$ with a symmetric monoidal structure reduces to see whether $\mathbf{GrDecomp}$ is a monoidal subcategory of $\mathbf{Funct}(\Delta^{op}, \mathbf{DiscGrpd}/\mathbb{Z})$ or not.

Proposition 4.4.3. *The subcategory*

$$\mathbf{GrDecomp} \subseteq (\mathbf{Funct}(\Delta^{op}, \mathbf{DiscGrpd}/\mathbb{Z}), \otimes, I, B)$$

is a monoidal subcategory.

Proof. We start off by checking that $\mathbf{GrDecomp}$ is closed under the tensor product $\otimes$ in the simplicial category. In order to see it, observe that it suffices to check that the tensor product of pullback cubes in $\mathbf{DiscGrpd}/\mathbb{Z}$ is also a pullback cube. But this is true thanks to lemma 3.3.2. Now the constant simplicial object

$$\begin{array}{rcl} I\colon \Delta^{op} & \longrightarrow & \mathbf{DiscGrpd}/\mathbb{Z} \\ c & \longmapsto & \begin{array}{c} 1 \\ \big\downarrow{\scriptstyle 0} \\ \mathbb{Z} \end{array} \\ c \longrightarrow c' & \longmapsto & \mathrm{id}_1, \end{array}$$

is trivially a graded discrete decomposition space, since identity cubes are always pullback cubes. Finally, we need to see that the associator, the left and the right unitors are CULF maps, but this is again obvious thanks to lemma 4.2.5. ∎

The interesting part of considering such a monoidal structure in $\mathbf{GrDecomp}$ is that it allows us to extend the functor

$$\Phi\colon \mathbf{GrDecomp} \longrightarrow \mathbf{Comon}(\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}))$$

to a functor

$$\Phi\colon \mathbf{Mon}(\mathbf{GrDecomp}) \longrightarrow \mathbf{Bimon}(\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}))$$

in a natural way.

Definition 4.4.4. A *monoidal decomposition space* is a monoid object (X, μ, η) in $(\mathbf{GrDecomp}, \otimes, I, B)$, and a *monoidal CULF map* is a monoid morphism in $(\mathbf{GrDecomp}, \otimes, I, B)$.

Let (X, μ, η) be a monoidal decomposition space. Then, we know that

$$\Phi(X) = (X_1 \xrightarrow{D_1} \mathbb{Z}, \delta_X, \varepsilon_X)$$

is a comonoid object in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$. Observe that the monoidal structure carried by X trivially makes $(X_1 \xrightarrow{D_1} \mathbb{Z}, \mu_1, \eta_1)$ into a monoid object in

$$\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}).$$

Let us quickly show why this is true by showing that the associative law holds, since the unit law is completely analogous. By hypothesis, $\mu\colon X \otimes X \longrightarrow X$ verifies the associative property, therefore we trivially have that so does

$$\mu_1 = \begin{tikzcd} X_1 \times X_1 \arrow[r, "\mu_1"] \arrow[d, "D_1+D_1"'] & X_1 \arrow[d, "D_1"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z}, \end{tikzcd}$$

and from this fact it is easy to see that so does the span

$$\Phi(\mu) = \begin{tikzcd} X_1 \times X_1 \arrow[r, equal] \arrow[d, "D_1+D_1"] & X_1 \times X_1 \arrow[r, "\mu_1"] \arrow[d, "D_1+D_1"] & X_1 \arrow[d, "D_1"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

Indeed, we have to show that the diagram

$$\begin{tikzcd} X_1 \otimes X_1 \arrow[r, "\Phi(\mu)"] & X_1 \\ X_1 \otimes X_1 \otimes X_1 \arrow[u, "\Phi(\mu)\otimes \mathrm{id}_{X_1}"] \arrow[r, "\mathrm{id}_{X_1} \otimes \Phi(\mu)"'] & X_1 \otimes X_1 \arrow[u, "\Phi(\mu)"'] \end{tikzcd} \tag{4.1}$$

commutes up to isomorphism of spans. If we expand this diagram

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_1 \times X_1 \arrow[ur, "2D_1"] \arrow[rr, equal] & & X_1 \times X_1 \arrow[ur, "2D_1"] \arrow[rr, "\mu_1"] & & X_1 \arrow[ur, "D_1"] & \\
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_1 \times X_1 \times X_1 \arrow[uu, "\mu_1 \times \mathrm{id}_{X_1}"] \arrow[ur, "3D_1"] \arrow[rr, equal] \arrow[dd, equal] & & X_1 \times X_1 \times X_1 \arrow[uu, "\mu_1 \times \mathrm{id}_{X_1}"] \arrow[ur, "3D_1"] \arrow[rr, "\mathrm{id}_{X_1} \times \mu_1"] \arrow[dd, equal] & & X_1 \times X_1 \arrow[uu, "\mu_1"] \arrow[ur, "2D_1"] \arrow[dd, equal] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_1 \times X_1 \times X_1 \arrow[ur, "3D_1"] \arrow[rr, equal] & & X_1 \times X_1 \times X_1 \arrow[ur, "3D_1"] \arrow[rr, "\mathrm{id}_{X_1} \times \mu_1"] & & X_1 \times X_1 \arrow[ur, "2D_1"] &
\end{tikzcd}
$$

we see that the indicated squares are trivially pullbacks, the bottom-left cube trivially commutes and the top-right cube commutes by hypothesis, therefore 4.1 commutes up to isomorphism of spans, as claimed. Thus, $X_1 \xrightarrow{D_1} \mathbb{Z}$ has both monoid and comonoid object structure. Furthermore, since the multiplication

$$\mu \colon X \otimes X \longrightarrow X$$

is CULF, we automatically have that

$$
\Phi(\mu) = \begin{tikzcd}
X_1 \times X_1 \arrow[r, equal] \arrow[d, "D_1+D_1"] & X_1 \times X_1 \arrow[r, "\mu_1"] \arrow[d, "D_1+D_1"] & X_1 \arrow[d, "D_1"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}
$$

is a coalgebra morphism, and the same happens with

$$
\Phi(\eta) = \begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "0"] & 1 \arrow[r, "\eta_1"] \arrow[d, "0"] & X_1 \arrow[d, "D_1"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}.
\end{tikzcd}
$$

Hence, $(X_1 \xrightarrow{D_1} \mathbb{Z}, \mu_1, \eta_1, \delta_X, \varepsilon_X)$ is a bimonoid object in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$. in order to complete the extension, it only remains to see that given a monoidal CULF map

$$(X, \mu^X, \eta^X) \xrightarrow{F} (Y, \mu^Y, \eta^Y),$$

the span

$$
\Phi(F) := \begin{tikzcd}
X_1 \arrow[r, equal] \arrow[d, "D_1^X"] & X_1 \arrow[r, "F_1"] \arrow[d, "D_1^X"] & Y_1 \arrow[d, "D_1^Y"] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}
$$

is a bimonoid morphism in $\overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z})$. We already know that it is a comonoid morphism, and it is a monoid morphism by hypothesis. Just in case, let us show at least that $\Phi(F)$ is multiplicative. This means that we have to show that the diagram

$$\begin{tikzcd}
X \otimes X \arrow[r, "\Phi(F)\otimes\Phi(F)"] \arrow[d, "\Phi(\mu^X)"'] & Y \otimes Y \arrow[d, "\Phi(\mu^Y)"] \\
X \arrow[r, "\Phi(F)"'] & Y
\end{tikzcd}$$

commutes up to isomorphism of spans. Expanding the diagram

$$\begin{tikzcd}
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_1 \times X_1 \arrow[ur, "2D_1^X"] \arrow[rr, equal] \arrow[dd, equal] & & X_1 \times X_1 \arrow[ur, "2D_1^X"] \arrow[rr, "F_1 \times F_1"] \arrow[dd, equal] & & Y_1 \times Y_1 \arrow[ur, "2D_1^Y"] \arrow[dd, equal] & \\
& \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_1 \times X_1 \arrow[ur, "2D_1^X"] \arrow[rr, equal] \arrow[dd, "\mu_1^X"'] & & X_1 \times X_1 \arrow[ur, "2D_1^X"] \arrow[rr, "F_1 \times F_1"] \arrow[dd, "\mu_1^X"] & & Y_1 \times Y_1 \arrow[ur, "2D_1^Y"] \arrow[dd, "\mu_1^Y"] & \\
& \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_1 \arrow[ur, "D_1^X"] \arrow[rr, equal] & & X_1 \arrow[ur, "D_1^X"] \arrow[rr, "F_1"] & & Y_1, \arrow[ur, "D_1^Y"] &
\end{tikzcd}$$

we can clearly see that our claim is true, since the indicated squares are pullbacks, the top-left square trivially commutes and the bottom-right square commutes precisely because F is a monoid morphism.

4.5 An Example: Interval Cell-Sets

Summing up, we have constructed a chain of functors

$$\mathbf{GrDecomp} \xrightarrow{\Phi} \overline{\mathbf{PSpan}}(\mathbf{DiscGrpd}/\mathbb{Z}) \xrightarrow{\Psi} \mathbf{Cell} \xrightarrow{Z_*} \mathbf{GAb}$$

that sends graded discrete decomposition spaces and monoidal decomposition spaces to coalgebras and bialgebras in $\mathbf{GAb}$, respectively. One of our objectives was to show that a large class of examples in [RS98] are in fact expressible in terms of decomposition spaces. However, for lack of time, we cannot achieve such a result. But what we can do is to give a detailed example of how a concrete cell-set can be reinterpreted in terms of decomposition spaces. We choose the example in 2.5.6, which we recall now: it describes the object cell-set $\mathscr{L}$ defined as all non-empty finite chains of a fixed poset P with chains of arbitrary length (for instance, $\mathbb{N}$ with the natural order) and dimension function given by the length of the chains. Our goal now is to construct a graded discrete groupoid that, via the functors we have defined, produces the same cell-set as $\mathscr{L}$. Recall that, since $\mathscr{L}$ is closed and connected, we know that the cell maps 2.5.2 and 2.3.8 are a coproduct and a counit for $\mathscr{L}$, respectively. This fact gives us the clue to define our candidate. Let X_0, X_1 and X_2 be the discrete groupoids over $\mathbb{Z}$ defined on objects as

$$X_0 := P,$$

$$X_1 := \Psi(\mathscr{L})^{-1},$$

$$X_2 := \{((a_1 < \cdots < a_{k+1}), i) : a_j \in P \text{ for every } 1 \leq j \leq k+1,\ 0 \leq i \leq k\},$$

and the arrows between objects are the unique poset isomorphisms between them. The dimension functors $D_i \colon X_i \longrightarrow \mathbb{Z}$, for $0 \leq i \leq 2$, are given by the length of the chains. Consider the arrows $d_0, d_1, d_2 \colon X_2 \longrightarrow X_1$ defined as follows:

$$d_0((a_1 < \cdots < a_{k+1}), i) := (a_{i+1} < \cdots < a_{k+1}),$$

$$d_1((a_1 < \cdots < a_{k+1}), i) := (a_1 < \cdots < a_{k+1}),$$

$$d_2((a_1 < \cdots < a_{k+1}), i) := (a_1 < \cdots < a_{i+1}).$$

Consider also the arrow

$$\begin{array}{rcl} s_0 \colon X_0 & \longrightarrow & X_1 \\ a & \longmapsto & s_0(a) := (a). \end{array}$$

Then, it should be clear that that the triple

$$(X_1 \xrightarrow{D_1} \mathbb{Z}, \delta_X, \varepsilon_X),$$

where

$$\delta_X = \begin{tikzcd} X_1 \arrow[d, "D_1"'] & X_2 \arrow[l, "d_1"'] \arrow[r, "(d_2,d_0)"] \arrow[d, "D_2"] & X_1 \times X_1 \arrow[d, "2D_1"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

and

$$\varepsilon_X = \begin{tikzcd} X_1 \arrow[d, "D_1"'] & X_0 \arrow[l, "s_0"'] \arrow[r, "t"] \arrow[d, "0"] & 1 \arrow[d, "0"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}, \end{tikzcd}$$

induces the coalgebra of spans that we want. First, it is trivial to see that the diagrams defined by δ_X and ε_X are indeed spans of discrete groupoids. Consider now

$$\Psi(X_1 \xrightarrow{D_1} \mathbb{Z}, \delta_X, \varepsilon_X) = (\mathscr{L}, \Psi(\delta_X), \Psi(\varepsilon_X)).$$

Then, for every $\underline{k} = (a_1 < \cdots < a_{k+1}) \in \mathscr{L}$, the coproduct is given by

$$\Psi(\delta_X)(\underline{k}) = \left\{(d_2, d_0)((\underline{l}, i)) : (\underline{l}, i) \in d_1^{-1}(\underline{k})\right\} =$$

$$\{((a_1 < \cdots < a_{i+1}), (a_{i+1} < \cdots < a_{k+1})) : 0 \leq i \leq k\},$$

and the counit is

$$\Psi(\varepsilon_X)(\underline{k}) = \left\{t(x) : x \in s_0^{-1}(\underline{k})\right\} = \begin{cases} \{*\}, & \text{if } |\underline{k}| = 1, \\ \varnothing, & \text{otherwise.} \end{cases}$$

This encourages us to keep defining the higher dimensional groupoids and the face and degeneracy maps between them. For $n \in \{0, 1, 2\}$ we already know what X_n means. For $n \geq 3$, define

$$X_n := \{((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-1}) : a_j \in P,\ 1 \leq j \leq k+1,\ 0 \leq i_1 \leq \cdots \leq i_{n-1} \leq k\},$$

which is a discrete groupoid with morphisms defined as before. Now we want to know the definition of the faces and degeneracies. Let us first define the faces $d_r \colon X_n \longrightarrow X_{n-1}$, for $0 \leq r \leq n$ and $n \geq 1$. If $n = 1$, then

$$d_0(a_1 < \cdots < a_{k+1}) = a_{k+1},$$

$$d_1(a_1 < \cdots < a_{k+1}) = a_1.$$

Now suppose that $n \geq 2$. If $r \neq 0, n$, then

$$d_r((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-1}) := ((a_1 < \cdots < a_{k+1}), i_1, \ldots, \hat{i_r}, \ldots, i_{n-1}),$$

where the notation $\hat{i_r}$ means, as usual, that the element i_r is missing. If $r = 0, n$, we define

$$d_0((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-1}) := ((a_{i_1+1} < \cdots < a_{k+1}), i_2 - i_1, \ldots, i_{n-1} - i_1)$$

and

$$d_n((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-1}) := ((a_1 < \cdots < a_{i_{n-1}+1}), i_1, \ldots, i_{n-2}).$$

Finally, we define the degeneracies $s_r \colon X_n \longrightarrow X_{n+1}$, for $0 \leq r \leq n$ and $n \geq 0$. They only act on indices, putting a repeated index in the right position. Let us be more precise with what we mean: first, recall that for $n = 0$ we already gave the definition. For $n = 1$, we define

$$s_0(a_1 < \cdots < a_{k+1}) := ((a_1 < \cdots < a_{k+1}), 0),$$

$$s_1(a_1 < \cdots < a_{k+1}) := ((a_1 < \cdots < a_{k+1}), k).$$

Now assume that $n \geq 2$. If $r \neq 0, n$, we define

$$s_r((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-1}) := ((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{r-1}, i_r, (i_r)_{r+1}, i_{r+1}, \ldots, i_{n-1}),$$

where $(i_r)_{r+1}$ means that the index i_r is in the position $r + 1$. If $r = 0, n$, then

$$s_0((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-1}) := ((a_1 < \cdots < a_{k+1}), 0, i_1, \ldots, i_{n-1})$$

and

$$s_n((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-1}) := ((a_1 < \cdots < a_{i_{k+1}}), i_1, \ldots, i_{n-1}, k).$$

Before showing that this construction produces a graded discrete decomposition space, let us introduce some notation. For every $n \geq 0$ and every

$$(\underline{k}, i_1, \dots, i_{n-1}), (\underline{l}, j_1, \dots, j_{n-1}) \in X_n,$$

we denote by $\alpha_{\underline{k},\underline{l}} \colon x \longrightarrow y$ the unique poset isomorphism between the chains $\underline{k}$ and $\underline{l}$.

Proposition 4.5.1. *The data defined above defines a graded discrete decomposition space* $X \colon \Delta^{op} \longrightarrow \mathbf{DiscGrpd}/\mathbb{Z}$*.*

Proof. First of all, the most fundamental thing we need to verify is that X defines, indeed, a simplicial object. Thanks to proposition 4.1.4, this can be done by checking the following list of cosimplicial identities:

1. $d_i \circ d_j = d_{j-1} \circ d_i$, if $i < j$,
2. $s_i \circ s_j = s_{j+1} \circ s_i$, if $i \leq j$,
3. $d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i, & \text{if } i < j, \\ \mathrm{id}, & \text{if } i = j \text{ or } i = j+1, \\ s_j \circ d_{i-1}, & \text{if } i > j+1. \end{cases}$

After several computations, one can verify that such relations hold. Now let us check the compatibility with the grading. For every n, the dimension functors $D_n \colon X_n \longrightarrow \mathbb{Z}$ must verify the following conditions:

1. $D_0(x) = 0$, for every $x \in X_0$,
2. $(D_{n+1} \circ s_j)(x) = D_n(x)$, for every $0 \leq j \leq n$ and for every $x \in X_n$,
3. $(D_{n-1} \circ d_i)(x) = D_n(x)$, for every $0 < i < n$ and for every $x \in X_n$,
4. $(D_{n-1} \circ d_0)(x) = D_n(x) - (D_1 \circ d_\top^{n-1})(x)$, for every $x \in X_n$,
5. $(D_{n-1} \circ d_n)(x) = D_n(x) - (D_1 \circ d_\perp^{n-1})(x)$, for every $x \in X_n$.

The first property is trivial, since points are zero dimensional. The second and the third ones are also trivial, since the indicated faces and degeneracies do not change the chain, they only act on indices. Now we focus on the fourth property. Let $x = ((a_1 < \cdots < a_{k+1}), i_1, \dots, i_{n-1}) = (\underline{k}, i_1, \dots, i_{n-1}) \in X_n$. If $n = 1$, the property trivially holds, so suppose that $n \geq 2$. The left-hand side in the fourth is

$$(D_{n-1} \circ d_0)(x) = k - i_1.$$

For the right-hand side, it is easy to check by induction that

$$d_\top^{n-1}(x) = (a_1 < \cdots < a_{i_1+1}).$$

Therefore,

$$D_n(x) - (D_1 \circ d_\top^{n-1})(x) = k - i_1.$$

The fifth property is completely analogous, so we omit the proof.

Now that we know that X is a simplicial object with the dimension property, it only remains to check that it takes generic-free pushouts to pullbacks. Thanks to proposition 4.2.3, we know that this is equivalent to ask that the following commutative cubes are pullbacks

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_1 \arrow[ur] \arrow[rr, "s_1"] \arrow[dd, "d_\perp"'] & & X_2 \arrow[ur] \arrow[dd, "d_\perp"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_0 \arrow[ur] \arrow[rr, "s_0"] & & X_1 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_1 \arrow[ur] \arrow[rr, "s_0"] \arrow[dd, "d_\top"'] & & X_2 \arrow[ur] \arrow[dd, "d_\top"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_0 \arrow[ur] \arrow[rr, "s_0"] & & X_1 \arrow[ur] &
\end{tikzcd}
\tag{4.2}
$$

and the following commutative cubes are pullbacks for every $n \geq 2$ and some $0 < r = r(n) < n$, depending on n

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_{n+1} \arrow[ur] \arrow[rr, "d_{r+1}"] \arrow[dd, "d_\perp"'] & & X_n \arrow[ur] \arrow[dd, "d_\perp"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_n \arrow[ur] \arrow[rr, "d_r"] & & X_{n-1} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, equal] \arrow[dd, equal] & & \mathbb{Z} \arrow[dd, equal] \\
X_{n+1} \arrow[ur] \arrow[rr, "d_r"] \arrow[dd, "d_\top"'] & & X_n \arrow[ur] \arrow[dd, "d_\top"] & \\
 & \mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \\
X_n \arrow[ur] \arrow[rr, "d_r"] & & X_{n-1}. \arrow[ur] &
\end{tikzcd}
\tag{4.3}
$$

Let us start with the first pair of cubes 4.2. We will only see that the left-hand side cube is a pullback; the proof for the right one is completely analogous. The standard model for the pullback of the cospan within the left-hand side cube is given, in objects, by

$$
X_0 \times_{X_1} X_2 = \{(a, (a_1 < \cdots < a_{k+1}, i)) \in X_0 \times X_2 : s_0(a) = d_\perp(a_1 < \cdots < a_{k+1}, i)\} =
$$

$$
\{(a, (a_1 < \cdots < a_{k+1}, i)) \in X_0 \times X_2 : (a) = (a_{i+1} < \cdots < a_{k+1})\} =
$$

$$
\{(a_{k+1}, (a_1 < \cdots < a_{k+1}, k)) \in X_0 \times X_2\}\,.
$$

Now for every $x = (a_{k+1}, (a_1 < \cdots < a_{k+1}, k)) = (a_{k+1}, (\underline{k}, k)), y = (b_{k+1}, (b_1 < \cdots < b_{k+1}, k)) = (a_{k+1}, (\underline{l}, k))$,

$$
X_0 \times_{X_1} X_2((a_{k+1}, (\underline{k}, k), (b_{k+1}, (\underline{l}, k)) = \left\{(\alpha_{a_{k+1}, b_{k+1}}, \alpha_{\underline{k}, \underline{l}})\right\}.
$$

Thus, there is an obvious span isomorphism

$$
\begin{tikzcd}
& & X_1 \arrow[dll, "d_\perp"'] \arrow[d] \arrow[drr, "s_1"] \arrow[dd, bend left, "\varphi"] & & \\
X_0 \arrow[d] & & \mathbb{Z} \arrow[dll, equal] \arrow[drr, equal] \arrow[dd, bend left, equal] & & X_2 \arrow[d] \\
\mathbb{Z} \arrow[drr, equal] & & X_0 \times_{X_1} X_2 \arrow[ull, "p_1"'] \arrow[urr, "p_2"] \arrow[d] & & \mathbb{Z} \arrow[dll, equal] \\
& & \mathbb{Z} & &
\end{tikzcd}
$$

where $\varphi\colon X_1 \longrightarrow X_1 \times_{X_0} X_2$ is the discrete groupoid isomorphism given by

$$
\begin{array}{rcl}
\varphi\colon X_1 & \longrightarrow & X_0 \times_{X_1} X_2 \\
(a_1 < \cdots < a_{k+1}) & \longmapsto & (a_{k+1}, (a_1 < \cdots < a_{k+1}, k)) \\
\alpha_{\underline{k},\underline{l}} & \longmapsto & (\alpha_{a_{k+1},b_{k+1}}, \alpha_{\underline{k},\underline{l}}),
\end{array}
$$

and p_1, p_2 are the canonical projections. Now for the remaining cubes in 4.3 we will do the same as before: we will show that the left-hand side cube is a pullback, letting the proof of the right one for the reader, which is completely analogous. The pullback of the cospan inside the left-hand side cube is the groupoid that, in objects, is given by

$$X_n \times_{X_{n-1}} X_n = \{(a,b) \in X_n \times X_n : d_r(a) = d_0(b)\}.$$

In fact, in this example the choice of r does not matter but, for simplicity, let us choose $r = r(n) = n-1$. Now given $(a,b) \in X_n \times X_n$, write

$$
\begin{cases}
a = ((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-1}), \\
b = ((b_1 < \cdots < b_{l+1}), j_1, \ldots, j_{n-1}).
\end{cases}
$$

Then, the condition

$$d_{n-1}(a) = d_0(b)$$

is equivalent to

$$((a_1 < \cdots < a_{k+1}), i_1, \ldots, i_{n-2}) = ((b_{j_1+1} < \cdots < b_{l+1}), j_2 - j_1, \ldots, j_{n-1} - j_1),$$

which is equivalent to ask that

$$
\begin{cases}
(a_1 < \cdots < a_{k+1}) = (b_{j_1+1} < \cdots < b_{l+1}), \\
i_1 = j_2 - j_1, \\
\vdots \\
i_{n-2} = j_{n-1} - j_1,
\end{cases}
$$

that is,

$$\begin{cases} j_1 = l - k, \\ a_1 = b_{l-k+1}, \\ \vdots \\ a_{k+1} = b_{l+1}, \\ i_1 = j_2 - j_1, \\ \vdots \\ i_{n-2} = j_{n-1} - j_1. \end{cases}$$

Thus,

$$X_n \times_{X_{n-1}} X_n = \{(a, b) \in X_n \times X_n\},$$

where

$$a = ((b_{l-k+1} < \cdots < b_{l+1}), j_2 - l + k, \ldots, j_{n-1} - l + k, i_{n-1})$$

and

$$b = ((b_1 < \cdots < b_{l+1}), l - k, j_2 \ldots, j_{n-1}).$$

The arrows are the corresponding pairs of isomorphisms, just as we did before. Letting p_1 and p_2 be the natural projections, we observe that there is a span isomorphism

$$\begin{tikzcd}
 & X_{n+1} \arrow[dl, "d_\perp"'] \arrow[dr, "d_{r+1}"] \arrow[d] \arrow[dd, bend left, "\varphi"] & \\
X_n \arrow[d] & \mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, bend left, equal] & X_n \arrow[d] \\
\mathbb{Z} \arrow[dr, equal] & X_n \times_{X_{n-1}} X_n \arrow[ul, "p_1"] \arrow[ur, "p_2"'] \arrow[d] & \mathbb{Z} \arrow[dl, equal] \\
 & \mathbb{Z} &
\end{tikzcd}$$

where $\varphi\colon X_{n+1} \longrightarrow X_n \times_{X_{n-1}} X_n$ is the discrete groupoid isomorphism given by

$$\begin{array}{rcl} \varphi\colon X_{n+1} & \longrightarrow & X_n \times_{X_{n-1}} X_n \\ b = ((b_1 < \cdots < b_{l+1}), j_1, \ldots, j_n) & \longmapsto & (\varphi_1(b), \varphi_2(b)), \end{array}$$

where

$$\varphi_1(b) = ((b_{j_1+1} < \cdots < b_{l+1}), j_2 - j_1, \ldots, j_{n-1} - j_1, j_n - j_1)$$

and

$$\varphi_2(b) = ((b_1 < \cdots < b_{l+1}), j_1, j_2, \ldots, j_{n-1}).$$

Its inverse is given by

$$\begin{array}{rcl} \varphi^{-1}\colon X_n \times_{X_{n-1}} X_n & \longrightarrow & : X_{n+1} \\ (a, b) = ((b_1 < \cdots < b_{l+1}), j_1, \ldots, j_n) & \longmapsto & \varphi_1^{-1}(a, b), \end{array}$$

where, if we write

$$\begin{cases} a = ((b_{j_1+1} < \cdots < b_{l+1}), j_2 - j_1, \ldots, j_{n-1} - j_1, i_{n-1}), \\ b = ((b_1 < \cdots < b_{l+1}), j_1, j_2 \ldots, j_{n-1}), \end{cases}$$

we have that

$$\varphi_1^{-1}(a,b) := (b_1 < \cdots < b_{k+1}, j_1, \ldots, j_{n-1}, i_{n-1} + j_1).$$

Thus, $X\colon \Delta^{op} \longrightarrow \mathbf{DiscGrpd}/\mathbb{Z}$ is a graded discrete decomposition space, as desired. $\blacksquare$

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